Bounding the Number of k-faces in Arrangements of Hyperplanes

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Abstract

An *arrangement* of hyperplanes in the d-dimensional space is a set of n hyperplanes $\mathcal{H}$ together with the associated cell decomposition of $\mathbb{R}^d$. In this paper, we deal with the enumeration of the face numbers of arrangements. $f_k(\mathcal{H})$ denotes the number of k-dimensional faces (the face number) of the arrangement $\mathcal{H}$. We first show the following relation between the k-dimensional face number and the (k+1)-dimensional face number:

$$(1) \quad f_{k+1}(\mathcal{H}) \geq \frac{d-k}{k+1} f_k(\mathcal{H}).$$

This inequality shows a property of the *incidence graph* representing the arrangement, and it is significant since it holds for arrangements with singularities. We can give similar formulas for arrangements of spheres in the d-dimensional sphere, arrangements in the d-dimensional projective space, and oriented matroids. As a corollary of (1), we obtain

$$(2) \quad f_k(\mathcal{H}) \leq \binom{d}{k} f_d(\mathcal{H}).$$

The later implies that the total number of faces in an arrangement is polynomially bounded by the number of maximal faces. This fact can be applied to the analysis of algorithms to construct arrangements from their underlying oriented matroids.

Another contribution of this paper is the discovery of *logarithmic concaveness* of both the *f-vector* – the sequence of face numbers of a cell decomposition – and its associated *h-vector* of a simple arrangement. Logarithmic concaveness of combinatorial sequences is a recent topic in enumerative combinatorics related to algebra and geometry, while *f-vectors* and *h-vectors* occur in many branches of mathematics. Our logarithmic concaveness theorem is one of the few known results on non-simplicial cell complexes.