General Metrics and Angle Restricted Voronoi Diagrams

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Abstract

This paper contains results about generalizations of metrics in the plane. First, we find that the MST \( \subseteq \) RNG \( \subseteq \) DT inclusion, already known for \( L_p \) metrics, holds for much more general metrics. Examples include the non translation invariant geodesic metrics defined from geodesics through weighted regions or around obstacles. Second, we return to translation invariance and present results based on angle restricted versions of ordinary metrics and related notions. Let \( VI_\alpha(p) \) denote the closed region between the two rays in directions \( r_1, r_2 \) from \( p \) forming an acute angle \( \alpha = [r_1, r_2] \). For any convex base metric \( d \), \( d_\alpha \) is defined by \( d_\alpha(x,y) = d(x,y) \) if \( y \in VI_\alpha(x) \) and \( d_\alpha(x,y) = \infty \) otherwise.

MST \( \subseteq \) DT for the Euclidean metric was shown by [Shamos and Hoey 75]. The RNG is defined so for \( x, y \in S \), \( (x,y) \in RNG \) iff for all \( z \in S - \{x,y\} \), \( d(x,y) \leq d(x,z) \) or \( d(x,y) \leq d(y,z) \). [Jaromczyk and Kowaluk 87] showed MST \( \subseteq \) RNG \( \subseteq \) DT for all \( L_p \) metrics. The proofs [Toussaint 80, Jaromczyk and Kowaluk 87] that MST \( \subseteq \) RNG apply to any symmetric weight function \( d \). We find that the RNG \( \subseteq \) DT inclusion also holds for any symmetric convex metrics that have geodesics in the following sense: For any \( x, y \) in the domain, there exists a continuous curve \( \gamma(x,y) \) between \( x \) and \( y \) such that for all \( z \in \gamma \), \( d(x,y) = d(x,z) + d(z,y) \).

We present an \( O(N\log N) \) divide and conquer algorithm for computing the Voronoi diagram of a finite set of \( N \) points with the distance function \( d_\alpha \). This algorithm implies, for any convex distance function, we can compute in \( O(N\log N) \) time the all pairs geographic nearest neighbors of a set of \( N \) sites in the plane. In [Yao 82], an \( O(N^{2-1/8}\log^{2-1/8}N) \) time algorithm for this problem is given for \( L_1, L_2 \) and \( L_\infty \) metrics. For \( L_1 \) and \( L_\infty \) metrics, an \( O(N\log N) \) time algorithm is given by [Guibas and Stolfi 83].

Our algorithm for computing the Voronoi diagram uses the divide and conquer strategy of [Lee 80, Chew and Drysdale 85] in which left and right Voronoi diagrams are merged. Our major modifications consist of a simplifying choice of the partition line angle and a fast new \( O(N) \) time procedure to calculate the upper end point of the bounded chain. We give a "forbidden region" theorem for all convex metrics that restrains the bisector for a pair of points. This derives monotonicity properties that help our algorithm analysis. It also suggests schemes for numerical approximation of the bisector curves with accuracy independent of the metric.

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