Polyhedra: Faces are Better than Vertices
(Extended Abstract)
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1. Introduction

A polyhedron is a closed surface that partitions points in $E^3$ into 3 sets: (i) Points lying inside the polyhedron, (ii) points incident on polyhedron and (iii) points outside the polyhedron. The boundary of the polyhedron is a two-manifold. Viewed combinatorially, the surface consists of faces, edges and vertices such that each edge is shared by exactly two faces and there is a single cycle of faces around each vertex such that each consecutive pair of faces has an edge in common. We only consider polyhedra with planar faces, that is, each face is contained in a plane. The number of holes in a polyhedron is its genus.

Several representation schemes have been proposed for the representation of polyhedra. Refer to [REQA80], [MANM88] for a detailed review on representations of polyhedra. Obtaining a polyhedron from its representaton is termed reconstruction. If there is only a single polyhedron that can be obtained from its representation, the polyhedron is uniquely reconstructible in that representation scheme. The problem we consider in this paper is the unique reconstruction of polyhedra in a representation scheme encoding the minimum amount of information.

The spherical dual representation [ROAJ86] of a polyhedron is a graph in which each face of the polyhedron is a node, and is labeled by the equation of the plane containing the face. A node is connected by an arc to another if the two faces the nodes represent share an edge in the polyhedron. No ordering of the arcs around each node is specified. (We use the non-standard terms nodes and arcs for elements of the graph to avoid confusing them with the vertices and edges of the polyhedron the graph represents.) The spherical dual representation, SDR, is a representation scheme of polyhedra that can be used in both solid modeling and computer vision. A variation on this scheme of representation was used in [ROAJ37]. The SDR provides some interesting relationships between the representation of an object and its image under perspective projection [PARP89].

Edmonds theorem [GROJ87] states that every rotation system for a graph $G$ induces (up to orientation-preserving equivalence of embeddings) a unique embedding of $G$ into an oriented surface. Most arguments on the representation of polyhedra are based on Edmonds theorem. Using Edmonds theorem, Weiler [WEIK85] shows that knowing the ordered set of edges around each vertex or each edge, or each face of a polyhedron is sufficient information for the unique reconstruction of any polyhedron. Weiler also states that a representation without order has insufficient information for unique reconstruction of a polyhedron. The domain of Edmonds theorem and Weiler's work includes polyhedra with non-planar faces.

The SDR can be viewed as the dual of the wire frame representation of a polyhedron. It is well known [REQA80], [MANM88], that the wire frame representation, i.e., vertex connectivity graph of a polyhedron is ambiguous. Algorithms for the unique reconstruction of a polyhedron, given its vertex or face connectivity graph, exist only if the polyhedron has genus 0 and the graph is three-connected [HANP82].
Even though SDR is a representation devoid of order information, we give an algorithm to
reconstruct any genus 0 polyhedra uniquely given its SDR. The algorithm has worst case complexity
$O(n^2)$ where $n$ is the size of the SDR. In addition we show that the SDR is not exactly the dual of
the wire frame representation in the context of ambiguity. This is accomplished by an example (the
4-dimensional hypercube) which is ambiguous as a vertex connectivity graph but is unambiguous as
an SDR. We then generalize this argument to prove that SDRs with at most degree 4 represent
polyhedra of arbitrary genus unambiguously. An $O(n)$ algorithm for unique reconstruction is also
given. These results are the first of their kind.

2. Related Work

By far the most important result on the realization of a polyhedron from its adjacency graph
structure is Steinitz’s theorem [GRUB67]. The theorem states that a graph $G$ is realizable as a convex
polyhedron if and only if $G$ is planar and triconnected. In the case of reconstruction of polyhedra,
this theorem can be used for all polyhedra that are combinatorially convex. A polyhedron is called
combinatorially convex if its vertex adjacency graph is planar and triconnected and the polyhedron
has genus 0.

Given a connected graph $G$, a closed surface $S$, and an embedding $i : G \to S$, a dual graph is
defined as follows. For each region $f$ of the embedding $i : G \to S$, place a node $f^*$ in its interior.
Then, for each arc $e$ of the graph $G$, draw an arc $e^*$ between the nodes just placed in the interiors
of the regions containing $e$. The resulting graph with nodes $f^*$ and arcs $e^*$ is called the (topological)
dual graph $G^*$ for the embedding $i : G \to S$. From Whitney, [GROJ87], a triconnected planar graph
has a unique embedding in the plane and hence a unique dual.

Based on these theorems, [HANP82] gives a linear time algorithm for the unique reconstruction
of a genus 0 polyhedron given its wire frame representation. This algorithm, however, requires the
wire frame input of the polyhedron to be three-connected and planar in the graph-theoretic sense.
The faces of the polyhedron correspond to the regions in the unique planar embedding of the vertex
connect graph. From Whitney’s theorem, every combinatorially convex polyhedron is uniquely
reconstructible from its dual. This dual is the underlying graph of SDR.

Markowsky and Wesley [MAR80] present an algorithm that generates all polyhedra with a
given wire frame. This explicitly uses topological and geometric information by forcing the final
faces to be planar. Human intervention is required to choose a polyhedron reconstructed from such
a wire frame representation.

3. Genus 0 Algorithm

Each face of a polyhedron consists of a connected region of a plane bounded by one or more
closed circuits of edges (straight line segments). The end points of an edge are the vertices of the
polyhedron. Each line segment is shared by exactly 2 faces of a polyhedron and a vertex is common
to at least 3 faces. It is possible for a pair of faces to share two or more edges. SDR does not
represent such multiple adjacency.

Let $SDR = (N, A)$ be the face connectivity graph of a polyhedron $P$. Consider a face $f_i$
of $P$ and its adjacent faces $A_i = \{f_1, f_2, ..., f_k\}$. The intersection of the planes containing $f$ and
$f_j$, $j = 1, ..., k$, defines an infinite line $l_j$. Any edge shared by $f_i$ and $f_j$ is a line segment in $l_j$. The
vertices of $P$ lying in $f_i$ are a subset of the points formed by the intersection of lines $l_1, l_2, \ldots, l_k$. There are $O(k^2)$ points of intersection.

Let $Q_i$ be these intersections. Similar points are found on each plane containing a face of $P$. Let $Q = \cup Q_i$. A subset of these points and line segments between them constitute the actual vertices and edges of $P$. Many of the points in $Q$ can be eliminated by use of the connection information in the SDR. For each point $q \in Q$, a sequence $S$ of faces defining $q$ is obtained from the intersections. Ordering in $S$ is obtained by traversing across edges with $q$ as an endpoint to an adjacent face until a cycle is complete. The definition of a polyhedron requires that there be only one circuit of polygons around a vertex. Thus for any point $p$ that is a vertex of $P$, there must correspond a simple circuit of nodes in the SDR that defines $p$ as a vertex. Accordingly, if there does not exist such a simple circuit, $q$ is deleted from $Q$.

Let $B_i$ be the set of faces forming all the points remaining in $Q_i$ and $SDR_i = (B_i, E_i)$ be the subgraph of SDR induced by $B_i$. $SDR_i$ contains all the information about face $f_i$ since all the faces connected to $f_i$ in both the vertex and edge sense are in $SDR_i$ along with all the arcs in between them. Corresponding to each vertex of $P$ in $f_i$, a simple circuit exists in $SDR_i$ but the converse is not true. Each face of a polyhedron has two adjacent faces at every vertex incident to it, hence every node of $SDR_i$ is at least of degree 2. The elimination of a node of degree two by merging two arcs in series does not affect embedding in a plane [GROJ87]. In general, no geometric information is lost since the deleted vertices of the graph are affinely dependent on the other vertices defining the region. Due to the merger of arcs in series $SDR_i$ may become a multigraph. Again from graph theory, we know that deletion of all but one of a set of parallel edges does not affect the embedding of the graph. No geometric information is lost since the parallel edges are only an artifact of the reduction process. We assume that $SDR_i$ has been subject to both series and parallel reduction.

**Observation 1:** If $SDR_i$ contains an articulation point, then $f_i$ is the articulation point.

**Observation 2:** If $SDR_i$ has a separation pair, $f_i$ is in the separation pair.

Geometrically, the presence of an articulation point in $SDR_i$ implies that the face $f_i$ has multiple connected boundaries. In genus 0 polyhedra, the articulation point is a necessary and sufficient condition for the existence of multiple connected boundaries. A separation pair does not uniquely specify any structure in general polyhedra. In genus 0 polyhedron, however, the separation pair is a necessary and sufficient condition for the two faces representing the nodes of the separation pair to define more than one edge of the polyhedron.

If the SDR is triconnected and planar, embedding it in a plane will uniquely determine the regions and their ordering around each node. Thus a unique reconstruction is possible. The SDR and $SDR_i$ of a genus 0 polyhedron are not necessarily triconnected. In general, the graphs have articulation points and separation pairs. Figure 1 shows a genus 0 polyhedron with an articulation point in its SDR. Finding the triconnected components of the SDR and embedding it in the plane will not work in general since some arcs are lost in decomposing the graph into triconnected components. Figure 2 shows such a polyhedron and its SDR, which on decomposition into triconnected components and embedding in the plane, defines some regions that do not correspond to any vertex of the input polyhedron. $SDR_i$, on the other hand, gives special status to face $f_i$ and retains all relevant nodes connected to $f_i$, assuring the integrity of the embedding. The algorithm broadly consists of the following steps.

1. For each face $f_i$, form the set $Q_i$. 

(2) Form the set $Q = \cup Q_i$.
(3) Delete elements of $Q$ that do not correspond to a circuit in $SDR$.
(4) For each face $f_i$ form the subgraph $SDR_i$ and reduce it.
(5) If $SDR_i$ has an articulation point, decompose the graph into two-connected components, retaining the articulation point in both the components.
(6) If the $SDR_i$ contains a separation pair, decompose the graph into three-connected components, retaining the separation pair in both components.
(7) Embed the three-connected components in the plane and find all the regions of which $f_i$ is a part, giving all vertices formed by face $f_i$.

The $SDR_i$ of each face of a genus 0 polyhedron was embedded in the plane independently to reconstruct that face of the polyhedron. In case of polyhedra of higher genus, this is not always possible. Spurious cycles are formed that cannot be detected locally. Information from other faces can be used to delete some arcs from the other $SDR_i$ until no more arcs can be deleted. Since each loop on a face without multiple connections forms a three-connected planar graph when only the connections involved in forming the loop are retained, the result of deleting arcs ideally results in obtaining a three-connected planar graph for each loop of each face. Due to the complexity of the $SDR_i$ of polyhedra of arbitrary genus, this reduction cannot be done for all polyhedra. The hypercube presented in the next section is an example of such an $SDR$. We prove that the hypercube represents a unique polyhedron.

4. The Hypercube

The 'hypercube' is the connectivity structure of the graph in Figure 3. It is the classic example of the ambiguity of the vertex connectivity graph (wire frame model) [MANM88]. The topological dual of this graph is isomorphic to itself. In contrast to the vertex connectivity representation, we show that the hypercube as an $SDR$ is unambiguous.

Each face of a polyhedron having the hypercube as $SDR$ is connected by an edge to four other faces. These four lines defined by the face adjacencies form two different quadrilaterals. Refer to Figure 4. Two vertices remain the same between the interpretations. One vertex $a$ which we call the fixed vertex has its context unchanged, i.e., when we follow the boundary of the two polygons in the same direction, the line segments occur in the same order. At the other vertex $b$, called the reflex vertex however, the context is reversed. The internal angle at the reflex vertex changes from being a convex angle in one case to concave in the other. It is immediately clear from Figure 4 that fixing any one of the remaining four vertices ($c, d, e, f$) determines the polygon unambiguously. Vertices on each of the lines through the fixed vertex and nearer the fixed vertex are called intruded vertices and those farther away are called extruded vertices. Thus in face 1 in Figure 4, vertices $c$ and $d$ are intruded vertices and $e$ and $f$ are extruded vertices.

For the hypercube to have two interpretations as polyhedra, some face must have two interpretations. By a long calculation, we prove the following lemma.

**Lemma 1:** There cannot exist a set of faces forming a reflex vertex such that every face has two interpretations.

Since only one face is sufficient to determine the polyhedron unambiguously, the following theorem immediately follows.

**Theorem 2:** The hypercube is unambiguous as an $SDR$. 

The following theorem can also be proved using Lemma 1.

**Theorem 3:** An SDR with at most degree four unambiguously represents a polyhedron of any genus. Moreover, there is a linear time algorithm to reconstruct such a polyhedron.

5. Conjecture

We conjecture that the SDR is an unambiguous representation for polyhedra of arbitrary genus.

6. References


Figure 1. A Polyhedron with an articulation point in its SDR
Figure 2. A Polyhedron and its incorrect 3-connected components
Figure 3. The hypercube connectivity graph
Figure 4. The two interpretations of face 1