Turning a Polygon Inside Out

William J. Lenhart
Dept. of Computer Science
Williams College

Sue H. Whitesides
School of Computer Science
McGill University

Extended Abstract

We present a simple necessary and sufficient condition for turning a polygon inside out. The vertices of the polygon, which are labeled, act like rotary joints. The edges are fixed-length links that are allowed to cross over one another as they move. To turn the polygon inside out means to convert it by a continuous motion in the plane to a polygon that is the mirror image (with respect to some arbitrary line in the plane) of the original one. Our simple necessary and sufficient condition is that the lengths of the second and third longest links sum to no more than half the perimeter of the polygon. We then use this condition to solve the following motion planning problem: given two configurations of a chain of links, the two endpoints of which are fixed, determine whether the chain can be moved from one configuration to the other. In fact, this can be determined in linear time. Furthermore, in the event of a yes answer, a sequence of simple motions achieving the reconfiguration can be computed in linear time.

1 Introduction

The problem of reconfiguring chains of links under various conditions has been considered in [1], [2] and [3]. In particular, these papers present polynomial time algorithms for motion planning problems that have an unbounded number of degrees of freedom. While there are general techniques for solving motion planning problems having a bounded number of degrees of freedom in polynomial time, problems having an unbounded number of degrees of freedom are often at least NP-complete. Hence it is interesting to see examples of motion problems that can be solved quickly despite having an unbounded number of degrees of freedom.

This paper contributes such an example.

Given a sequence \((x, y, z)\) of three non-collinear points in the plane, we call the sequence a left turn if it determines a counterclockwise cycle; otherwise we call the sequence a right turn. We say that two sequences \((x, y, z)\) and \((x', y', z')\) have opposite orientations if one is a left turn and the other is a right turn; otherwise, we say they have the same orientation.

A chain is a sequence of \(n\) links, \(L_1, \ldots, L_n\), connected by joints. \(L_i\) has joints \(v_{i-1}\) and \(v_i\) and length \(l_i\). Sometimes \(L_i\) will be denoted \([v_{i-1}, v_i]\). Each link can rotate freely about its joints. A configuration of a chain \(L_1, \ldots, L_n\) is a polygonal curve (possibly self-intersecting) that consists of \(n\) consecutive links of lengths \(l_1, \ldots, l_n\), respectively. A closed chain is chain such that \(v_0\) and \(v_n\) are the same joint. Hence a configuration of a closed chain is just a closed polygonal curve. An arm is a chain in which a fixed location in the plane has been associated
with \( v_0 \).

We use the term *linkage*, often denoted \( L \), to refer to a chain, a closed chain or an arm.

**Definition 1.1** Two configurations of a linkage \( L \) are equivalent if one configuration can be continuously transformed to the other in such a way that the lengths of the links remain fixed throughout the motion.

In mathematical terms, this means that there is a homotopy between the two curves having the property that the link lengths are preserved throughout the homotopy. If two configurations of \( L \) are equivalent, we say that \( L \) can be moved from the one configuration to the other.

**Definition 1.2** A configuration of a closed chain \( L \) is invertible if it is equivalent to its mirror image (with respect to some arbitrary line).

Since a simple polygon can be viewed as a configuration of a closed chain, we will refer to a polygon as invertible if it is equivalent to some mirror image of itself. Figure 1 shows a polygon that is invertible.

**Key idea:** Let \( L_i = [v_{i-1}, v_i] \) be a link of a closed chain \( L \). Suppose that some configuration of \( L \) can be moved to its mirror image with respect to some arbitrary line. Let \( v_j \) be any joint of \( L \) that is not collinear with \( L_i \) in the original configuration. Then at some moment during the motion, \( v_j \) lies on the line through \( L_i \) at that moment. This is because the orientation of \((v_j, v_{i-1}, v_i)\) in the original configuration of \( L \) differs from its orientation in any mirror image of that configuration. By determining what conditions on the lengths of the links allow each \( v_j \) to become collinear with each \( L_i \), we can establish a necessary condition for invertibility; as it happens, this condition is also sufficient.

![Figure 1: A polygon being inverted. \([v_0, v_1]\) rotates c.w. about \( v_0 \) as \([v_2, v_3]\) rotates first c.w., then c.c.w. about \( v_3 \).](image)

**Definition 1.3** Given a configuration of a closed chain \( L \), a joint \( v \) of \( L \) is said to be extensible if the configuration is equivalent to a configuration in which the angle between the two links adjacent to \( v \) is 180°. (When the angle of a joint equals 180°, we say it is extended.) Similarly, \( v \) is collapsible if the configuration is equivalent to one in which the angle is 0°. (If a joint has an angle of 0° we call it collapsed.)

### 2 Inverting a Polygon

One of our main results is the following:

**Theorem 2.1** Given a configuration of a closed chain \( L \), the sum of whose link lengths equals \( m \), the configuration can be inverted iff
the lengths of its second and third longest links sum to no more than \( m/2 \).

One consequence of this theorem is that invertibility is seen not to be a property of a particular configuration of a closed chain, but rather a property of the chain itself. That is, if some configuration of \( L \) can be inverted, then every configuration of \( L \) can be inverted.

**Lemma 2.1** Any configuration of a closed chain that forms a simple polygon is equivalent to some configuration that forms a convex polygon.

This lemma has the following consequence. For a given closed chain \( L \), either all its configurations are equivalent, or they partition into two classes. One class includes the convex polygons with clockwise orientation obtainable from \( L \) (vertices are visited in order \( v_0, v_1, \ldots \) to determine the orientation); the other class contains the convex polygons with the opposite orientation.

A configuration of a linkage is said to be flat if all of its links are collinear.

**Lemma 2.2** If a closed chain \( L \) admits a flat configuration, then some configuration of \( L \) in the form of a convex polygon is invertible.

Another useful lemma is the following:

**Lemma 2.3** If a configuration of a closed chain \( L \) contains a joint that can be collapsed but not extended, then that configuration is invertible.

**Key idea.** Since the chain has at least four links, by suitably choosing which joint to label \( v_0 \), we can assume that \( L_i, L_j \) and \( L_k \) are the three longest links, that \( L_j \) and \( L_k \) do not have a joint in common, and that \( i < j < k \). (\( L_i, L_j, \) and \( L_k \) do not necessarily appear in order of decreasing length.) See Figure 2.

![Figure 2: A polygon with longest links \( L_i, L_j \) and \( L_k \).](image)

If the condition is violated, we have

\[
\begin{align*}
    l_i + l_j & > m/2 \\
    l_i + l_k & > m/2 \\
    l_j + l_k & > m/2
\end{align*}
\]

Partition \( L \) into three chains as follows: the link \( L_i = [v_{i-1}, v_i] \), the 'left chain' \( (v_i, v_{i+1}, \ldots, v_j) \), and the 'right chain' \( (v_{i-1}, v_{i-2}, \ldots, v_j) \). Also, let \( s_j \) be the total length of the left chain not including link \( L_j \) and let \( s_k \) be the total length of the right chain not including link \( L_k \). Thus \( m = l_i + l_j + s_j + l_k + s_k \).

For \( v_j \) to lie on the line through \( L_i \), it can be shown that at least one of the following inequalities must be satisfied (these inequalities...
correspond to $v_j$ being on, to the left of, or to the right of $L_i$, respectively).

\begin{align*}
(l_j - s_j) + (l_k - s_k) & \leq l_i \\
(l_j - s_j) + l_i & \leq (l_k + s_k) \\
(l_k - s_k) + l_i & \leq (l_j + s_j)
\end{align*}

(2)

However, each of these inequalities contradicts one of the three inequalities in (1).

Now we give the idea behind the proof of the sufficiency of the condition. First, we show that if there exists a joint $v_i$ that neither extends nor collapses then we can find three links (two of which are $L_i$ and $L_{i+1}$) such that each pair of links has total length greater than half of the perimeter of $L$. This clearly violates the condition. We then prove by induction on the number of links in $L$ that the condition is sufficient by demonstrating that either the configuration of $L$ has a joint that collapses but does not extend (in which case we are done by Lemma 2.3), or the configuration of $L$ is equivalent to a configuration in which one of the joints of the longest link is extended. This new configuration can be viewed as a closed chain having one fewer link (the extended joint is 'frozen', making a new longest link), and the condition of the theorem still holds.

The main result is:

**Theorem 3.1** Given two configurations of a chain $C$ that has both endpoints fixed at particular points in the plane, it can be determined in linear time whether the two configurations are equivalent. Moreover, if the configurations are equivalent, it is possible to compute in linear time a sequence of simple motions that moves one of the configurations to the other.

### References


### 3 Application

We give an application to motion planning. From [1] we use:

**Definition 3.1** A simple motion of a linkage is a continuous motion such that

1. At most four joint angles are changing. (If the linkage is an arm, then one of the angles may be the angle formed by $l_1$ and some line through $v_0$.)

2. Each angle is changing monotonically.