The Minimum Cone-Segment Cover Problem

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Abstract

Given a source of X-rays emitting in a 'conic' fashion, one would like to construct a safety barricade that could block any accidental high emittance of radiation. The barricage consists of panels coated with lead to be placed within the cone of emission. Due to space constraint these panels can be put only along specific lines. If one wishes to minimize the total surface of the panels, then its argued that one panel is not always the best solution.

We formalize the problem as follows: let $E^2$ be the Euclidean plane, $O$ its origin. Let $C$ be a cone of $E^2$ with apex $O$, and bounding rays $r$ and $r'$. A set of segments $S$ subtend $C$ if the endpoints of the segments of $S$ lie on $r$ and $r'$. Let $L$ be a line intersecting $r$ and $r'$ such that $O$ and $S$ belong to the bounded subset of $E^2$ determined by $L$, $r$, and $r'$. Let $A$ be the unbounded subset of $E^2$ also determined by $L$, $r$, and $r'$. A minimum length-cover of $O$ given $C$ and $S$, is a set of sub-segments of $S$ with minimum length which hides $O$ from $A$.

We exhibit an optimal $\Theta(n \log n)$ algorithm for finding such a minimum cover when the edges of $E$ are non-intersecting, and an $O(m \log m + mn \log n)$ algorithm when the number of intersection points among the edges of $S$ is $m$.

1 Definitions

Let $h_0$ and $h_1$ be two closed half-planes whose respective bounding lines $l_0$ and $l_1$ intersect in $O$. The boundary of the intersection region $I$ of $h_0$ and $h_1$ consists of two closed rays $r_0$ and $r_1$, such that $r_0$ lies on $l_0$ and $r_1$ lies on $l_1$, and $r_0$ and $r_1$ share the common endpoint $O$. With respect to $O$, the ray of $\{r_0, r_1\}$ bounding $I$, in the clockwise direction from $I$ shall be called the right bounding ray. The other ray shall be called the left bounding ray. Without loss of generality, we shall assume that $r_0$ is the left ray and $r_1$ the right ray. We shall refer to the region $I$ as the cone $K = (r_0, r_1)$ determined by $r_0$ and $r_1$. In general, any two rays sharing the origin as their common endpoint and not contained in a common line define a unique cone.

Given a cone $C = (r, r')$, the angle of $C$ is defined to be the angle $\angle rOr'$, formed by $r$, $O$, and $r'$. Observe that this angle must always be less than $\pi$. The apz of any cone is the origin $O$. Given two cones $C$ and $C'$, if $C' \subseteq C$ then $C'$ shall be said to be a sub-cone of $C$. Two cones $C$ and $C'$ will be considered disjoint if the apex $O$ is the only point that belongs to both cones, contiguous if they intersect in a single ray, and overlapping otherwise.

Let $C$ be a cone in $E^2$ with apex $O$ and bounding rays $r$ and $r'$. The bisector of the cone $C$ is assumed to lie on the positive $x$-axis; all other cases are treated in a similar manner with a minor re-parameterization. Let $C'$ be the symmetric cone of $C$ with respect to $O$. Let $Q$ be the set of points between $C$ and $C'$ with positive $y$-coordinate, $Q'$ the set of points between $C$ and $C'$ with negative $y$-coordinate.

Let $E$ be a set of edges subtending $C$. Given an edge $e$ of $E$, a continuous interval of $e$ is called a segment of $e$. Let $e$ and $e'$ be two subtending edges over cone $C$. If the segment of $e$ subtending an arbitrary sub-cone $C'$ of $C$, is shorter than the corresponding segment of $e'$, then $e$ dominates $e'$. If none of the edges dominates the other, they are said to be incomparable. In the latter case, the minimum cover determined by these two edges is composed of two segments, each of which belongs to one of the edges. The ray separating the two segments is called the splitting ray of $e$ and $e'$.
2 The non-intersecting case

2.1 Linear Upper Bound

Let \( C \) be a cone with two subbending edges \( e \) and \( e' \)
with \( e \) dominating \( e' \). What happens if we 'expand' the
cone and the edges? Is the relationship between
\( e \) and \( e' \) preserved?

**Lemma 2.1** Let \( C \) be a cone divided into three contiguos sub-cones \( C_L, C_M \) and \( C_R \), termed left, middle and right sub-cones. Let \( e_1 \) and \( e_2 \) be two subbending edges of \( C \). Denote by \( e_{i,j} \) the segment of
edge \( e_i \) subbending sub-cone \( C_j \). If \( e_{i,M} \) dominates \( e_{j,M}, \ i \neq j \), then either \( e_{i,L} \) dominates \( e_{j,L} \), or \( e_{i,R} \) dominates \( e_{j,R} \).

**Proof** Outline of the proof: We express the length of \( e_{i,2} \) in terms of \( e_{i,1}, \) where \( J = L, R \); the choice
of \( J \) depends on the sign of the slopes of \( e_1 \) and \( e_2 \),
and the location of the intersection point of their underlyng lines in any of the two quadrants \( Q \) or \( Q' \).
For instance, if both edges are positively sloped and
the intersection point is in \( Q' \), and we assume that
\( e_{1,M} \) dominates \( e_{2,M}, \) then \( J = R \).

Let \( L(e_{i,M}) \) be the length of \( e_{i,M} \). We obtain the following relation:
\[
L(e_{i,M}) = W_i L(e_{i,J})
\]

Where \( W_i > 0 \). By hypothesis \( e_{i,M} \) dominates \( e_{j,M} \), thus \( L(e_{i,M}) < L(e_{j,M}) \). Then it is sufficient to prove that \( W_i > W_j \). Which is straightforward to establish.

If both \( e_{i,M} \) and \( e_{i,L} \) are dominant, we say that
\( e_{i,M} \) is left extensible, similarly if \( e_{i,M} \) and \( e_{i,R} \) are both dominant, \( e_{i,M} \) is right extensible.

If we know that \( e_{i,L} \) and \( e_{i,R} \) are dominant over their respective sub-cones, what can we say about
\( e_{i,M} \) ?

Define the Hull of two cones to be the smallest cone
containing them.

**Lemma 2.2** Let \( C_1 \) and \( C_2 \) be two disjoint cones, \( H \) their hull. Let \( e_1 \) and \( e_2 \) two subbending edges of \( H \), such that \( e_{1,i} \) dominates \( e_{2,i}, \ i = 1, 2 \), then \( e_1 \)
dominates \( e_2 \).

**Proof** The proof is by contradiction. It assumes that \( e_1 \) and \( e_2 \) are incomparable over \( H \), and uses the previous lemma to obtain a contradiction by noting that if the segment of \( e_2 \) over \( H \setminus (C_1 \cup C_2) \) is dominant, then it would be either left or right extensible, contradicting the dominance of \( e_{1,J} \), \( J = L, R \).

**Theorem 2.3** Given a cone \( C \) and a set of \( n \) non-intersecting subbending edges, the number of segments in any minimum cover, is no more than \( n \).

**Proof** By contradiction. Assume that there exists a minimum cone-segment cover solution \( M \) with \( m \) segments, \( m > n \). By the pigeon-hole principle, there exists at least two disjoint dominant segments over cones \( C_j, C_k \) that belong to the same edge \( e_i \). Let \( e_{i,4} \) be a minimal segment between \( e_{i,j} \) and \( e_{i,k} \). Since both segments \( e_{i,j} \) and \( e_{i,k} \) are dominant, the segment of \( e_i \) subbending \( \text{hull}(C_j, C_k) \) is also dominant by lemma 2.2, contradicting the fact that \( e_{i,4} \) is a dominant segment. Hence each subbending edge can contribute with at most one dominant segment. This fact establishes the result.

This upper bound is tight as established by theorem 2.6.

2.2 Outline of the Algorithm

The algorithm sorts the set \( E \) closest to furthest with
respect to the origin. Next, the set \( E \) is divided into
two non-intersecting subsets \( E_- \) and \( E_+ \), where \( E_- \)
contains all negatively sloped edges, and \( E_+ \) all posi-
tively sloped ones. Two distinct minimum-covers, \( M_- \) and \( M_+ \), are constructed in a symmetric way, one
with \( E_- \) as the set of subbending edges, the other
with \( E_+ \). The minimum-cover induced by \( E \) is con-
structed by means of a simple merge-like technique,
where segments of \( M_- \) and \( M_+ \) that cover a common
area of the cone are compared against each other; the
outcome of the comparison being that either one of
the two segments is dominant, in which case the other
one is discarded, or that they are incomparable, in
which case each of the two segments contributes with
one subsegment to the minimum cover \( M \). This pro-
cess is performed in one sweep with no backtracking.

Remains to show how to construct \( M_- \) and \( M_+ \).
Since the construction is symmetric, we will outline
the construction of \( M_+ \).

The first step removes all the edges of \( E_+ \) whose
underlying line intersects with the underlying line of
\( e_1 \) in \( Q' \), where \( e_1 \) is the closest edge of \( E_+ \) to \( O \). The
reason being that those edges are dominated by \( e_1 \).
This can be seen by using triangular inequalities in a
straightforward manner.

Let \( E'_+ = \{ e'_1, \ldots, e'_t \} \), be the ordered set of remaining
edges of \( E_+ \). The second step constructs \( M_+ \) as follows:
starting with \( e'_1 \) and \( e'_2 \), it computes their splitting ray \( p_{1,2} \). If \( p_{1,2} \) falls to the left of \( C \), \( e'_2 \)
is removed, if it falls to the right of \( C \), \( e'_1 \) is removed,
otherwise the segment of \( e'_1 \) to the right of \( p_{1,2} \) and
the segment of \( e'_2 \) to the left of \( \rho_{1,2} \) are inserted in \( M_+ \). The same process is repeated with \( e'_2 \) and \( e'_3 \). If \( e'_3 \) is removed, we move on to \( e'_4 \); if \( e'_2 \) is removed, we move back to \( e'_1 \). Otherwise, the segment of \( e'_2 \) is replaced by a two segments, one from \( e'_2 \), the other from \( e'_3 \). At the end of this step, we obtain a minimum cover that forms a “diagonal”, meaning that the segments of \( M_+ \) are monotone when viewed radially around \( O \). Each time the algorithm backtracks while processing this step it removes one edge. Thus constructing \( M_+ \) is done in linear time.

The following two theorems establish the correctness of the previous step.

Assume that the line equation of a subtending edge \( e_i \) is given by \( y = a_i x + b_i \). Furthermore,

- Let \( \delta_i = b_i/\sqrt{1 + a_i^2} \).
- Let \( \alpha = \sqrt{\frac{\delta_1}{\delta_2}} \).
- Let \( \tau_0 = \frac{a_1 - a_2 \alpha}{1 - \alpha} \).
- Let \( \tau_1 = \frac{a_1 + a_2 \alpha}{1 + \alpha} \).

Let \( y = t_1 x \) be the line equation of ray \( r_1 \); ray \( r' \) being its symmetric with respect to the \( x \)-axis has line equation \( y = -t_1 x \).

**Theorem 2.4** There exist a splitting ray if and only if \( \tau_0 \) belongs to the open interval \( ]-t_1, t_1[ \). Moreover, the line equation of the splitting ray is given by \( y = \tau_0 x \).

**PROOF** Let \( y = tx \) be the line equation of the splitting ray, \( r_1 \). Set \( C_1 = (r_1, r_1) \) and \( C_2 = (r_1, r') \). We want to fix \( t \) such that the sum of the lengths of \( e_1 \) and \( e_2 \) is minimized, or such that the sum of the lengths of \( e_{1,2} \) and \( e_{1,2} \) is minimized. Let \( f(t) = L(e_{1,1}) + L(e_{2,2}) \). The slope of the splitting ray is a root of \( f \). Deriving \( f \), and solving \( f'(t) = 0 \), yields \( \tau_0 \) and \( \tau_1 \) as roots. Further algebraic manipulations establish the result. \( \square \)

Assume that the splitting ray is given by \( y = \tau_i x \), \( i = 1, 2 \). Let \( S_1 = (r_1, r_1) \) and \( S_2 = (r_1, r_0) \) be respectively the left and right sub-cones, determined by \( y = \tau i x \). Let \( p \) be the intersection point of the underlying lines of \( e_1 \) and \( e_2 \).

**Theorem 2.5** If \( p \) belongs to \( Q \) and there exist a splitting ray, then the pair \( \{ e_{1,2}, e_{2,1} \} \) forms a minimum cover.

**PROOF** Outline of the proof: We show that \( L(e_{1,2}) < L(e_{2,1}) \) by simple algebraic manipulations of the expressions of \( L(e_{i,j}) \).

The proof of correctness has been sketched by the various lemmas and theorems stated. The overall idea being that no edge can participate with more than one segment in the minimum solution. This result lead to the computation of the unique splitting ray between two subtending edges. By studying the relationships between the slope of the splitting rays and the slope of the edges, and the location of the intersection point of their underlying line, minimum pairs where determined in the previous theorem. The generalization of this result to \( n \) edges lead correctly to the outlined algorithm.

Aside from the initial sorting step, every other step is performed in linear time: dividing \( E \) into two sets, eliminating all the edges whose underlying lines intersect with the underlying line of \( e_1 \) in \( Q' \), constructing the sets \( M_+ \) and \( M_- \), and finally constructing the minimum cover \( M \).

### 2.3 Lower Bounds

Although the slopes of the splitting rays implies the use of the square root function, the algorithm does without it since its decisions are based on comparisons of slopes. Thus squaring the slopes of the splitting rays, a constant number of times, avoids using the square root. Moreover, the algorithm outputs a sequence of triplets of the form \((e_i, e_j, e_k)\), each of which represents a segment in the minimum cover, determined by the splitting rays of \((e_i, e_j)\) and \((e_j, e_k)\). Computing these splitting rays involves some implementation of the square root function; a step that is left open to the user.

The problem of determining this sequence of triplets has an \( \Omega(n \log n) \) lower bound, as established by the following two theorems.

**Theorem 2.6** Given a cone \( C \) and three non-intersecting positively sloped subtending edges \( e_1, e_2, e_3 \), with underlying lines intersecting at point \( p \) in \( Q \), then there exist a unique minimum cover containing one segment from each edge.

**PROOF** Outline of the proof: Assuming that the three edges are in order \( e_1, e_2, e_3 \), then by using implicit derivation, we show that the slope of the splitting ray of \( e_1 \) and \( e_2 \) is less than that of \( e_2 \) and \( e_3 \). Theorem 2.5 implies then that there are three segments in any derived minimum cover. Since the slopes of the splitting rays are fixed, there is but one minimum-cover.
Theorem 2.7 Finding the sequence of triplets requires in the worst case \( n \log n \) time under the algebraic model of computation.

**Proof** Transformation from the SUCCESSOR problem, which is defined as follows: Given a set \( A \) of \( n \) positive integers, determine the successor for each of them in \( A \). This problem has an obvious \( n \log n \) lower bound.

Map each integer \( a_i \) of \( A \) onto a line segment \( y = a_i x + b_i \), such that all line segments pass through a point \( p_i \) in \( Q \). Solve the Minimum Cone Segment Cover problem. By the previous theorem, any solution will contain \( n \) segments, which will be given in the form of a sequence of triplets. In linear time extract from each triplet the successor of the point, who was mapped to the middle line equation in the triplet. \( \square \)

3 The intersecting case

The algorithm computes all intersection points among the edges of \( E \) as follows: it sorts separately the set of vertices \( V_r \) lying on ray \( r \), and the set of vertices \( V_{r'} \) lying on ray \( r' \), based on the distance of the vertices from \( O \).

Let \( e_{i,j} = (v_{r,i}, v_{r,j}) \) be an edge of \( E \), with vertex \( v_{r,i} \) lying on \( r \) having rank \( i \) in the sorting of \( V_r; v_{r,j} \), is defined in a similar way.

The algorithm intersects \( e_{i,j} \) with \( e_{k,j} \), \( i = 1, \ldots, n, k = 1, \ldots, j - 1 \). Let \( T \) be the computed set of intersection points. The set \( T \) is then sorted radially around \( O \). Let \( q_1 \) be the first intersection point in the sorted order. The algorithm constructs the sub-cone \( C_1 \) of \( C \), determined by \( r \) and the ray \( r_1 \) passing through \( q_1 \). It then intersects \( C_1 \) with \( E \), and obtain the set \( E_1 \) consisting of non-intersecting segments. The next step is a call to the algorithm for the non-intersecting case with \( C_1 \) and \( E_1 \), which outputs a minimum cover for \( O \) given \( C_1 \) and \( E_1 \). The process is repeated over each consecutive sub-cone. After processing the last sub-cone, the minimum covers for the various sub-cones are merged to form the minimum cover for \( O \), given \( C \) and \( E \).

The correctness of this divide-and-conquer type of algorithm is based on the correctness of the algorithm that solves the non-intersecting case and on the following theorem.

**Theorem 3.1** Let \( M_1 \) and \( M_2 \) be two minimum covers for \( O \), given \( E_1 \) and \( C_i \) as defined above, \( i = 1, 2 \). Let \( C_{1,2} = C_1 \cup C_2 \) and \( E_{1,2} = E_1 \cup E_2 \). Denote by \( M_{1,2} \) the minimum cover for \( O \) given \( C_{1,2} \) and \( E_{1,2} \), then \( M_{1,2} = M_1 \cup M_2 \).

**Proof** Immediate, by contradiction. \( \square \)

The time analysis of this algorithm is straightforward: finding all intersection points requires \( O(m) \), where \( m \) is the number of intersection points. This is due to the fact that if \( e_{i,j} \) intersects with \( e_{k,j} \), then \( (v_{r,i}, v_{r,j}) \) has the reverse order of \( (v_{r'}, i, v_{r'}, j) \). Sorting the set \( Q \) takes \( m \log m \), solving the problem over a given sub-cone takes \( O(n \log n) \). Finally merging all the contiguous minimum covers requires \( O(m \cdot n) \).

References


