On Polygons Enclosing Point Sets

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Abstract
Let P and Q be collections of n and m points respectively, on the plane in general position. We say that P encloses Q if there is a closed simple polygon C with vertex set P such that all the elements of Q lie in the interior of C. Clearly if the elements of Q are not contained in the convex hull of P, Q cannot be enclosed by P. In this paper we prove that if Q is contained in the convex hull of P then P encloses at least half of the points of Q, and we will give examples to show that this bound is asymptotically tight. We also prove that if the polygon defined by the convex hull of P has at least m \((2 \log(m) + 1)\) vertices then P encloses all the m points of Q.

1. Introduction

Let \(P_n\) and \(Q_m\) be collections of n and m points on the plane such that \(Q_m\) is contained in the interior of the convex hull, \(\text{Conv}(P_n)\), of \(P_n\). We say that \(P_n\) encloses \(Q_m\) if there is a simple polygon C whose vertex set is exactly \(P_n\) such that \(Q_m\) is contained in the interior of C (See Figure 1(a).) For example if \(\text{Conv}(P_n) = k < n\) a collection \(Q_k\) not enclosable by \(P_n\) can be obtained by placing for each edge \(e\) of \(\text{Conv}(P_n)\) a point \(p_e\) in the interior of \(\text{Conv}(P_n)\) at a distance \(\varepsilon\) of the mid point of \(e\), where \(\varepsilon\) small enough (see Figure 1(b).)
It is clear from the example in Figure 1(b) that the condition of \( Q_m \) being contained in the interior of \( \text{Conv}(P_n) \) is not sufficient to guarantee that it is enclosable by \( P_n \). It is thus natural to ask the following: Given any two collections \( P_n \) and \( Q_m \) of points such that \( \text{Conv}(P_n) \supseteq Q_m \), is there a large subset \( H \) of \( Q_m \) that is enclosable by \( P_n \)?

Throughout this paper we will denote by \( P\text{Conv}(P_n) \) the polygon defined by the convex hull of \( P_n \), \( \text{Conv}(P_n) \), and a polygon whose vertex set is \( P_n \) will be called a \( P_n \)-polygon.

**Theorem 1:** Given any two collections of points \( P_n \) and \( Q_m \) such that \( \text{Conv}(P_n) \supseteq Q_m \) there is a \( P_n \)-polygon that encloses at least half of the points of \( Q_m \).

Our result follows immediately from the next lemma which is interesting on its own:

**Lemma 1:** Let \( P_n \) be any collection of points. Then there are two \( P_n \)-polygons whose union covers entirely \( \text{Conv}(P_n) \).

**Proof:** Let \( e \) be an edge of \( \text{Conv}(P_n) \) with end points \( u \) and \( v \) and let \( p_e \) be the mid-point of \( e \). Sort the points of \( P_n \) in the clockwise direction with respect to \( p_e \) and relabel them \( u=p_1,...,p_n=v \) accordingly (See figure 2(a).) Let \( S \) be the subsequence of \( u=p_1,...,p_n=v \) defined by \( [P_n-\text{Conv}(P_n)] \cup \{u,v\} \). Let \( \Phi_1 \) be the \( P_n \)-polygon \( p_1,...,p_n,p_1 \) (See figure 2(a).)
We define a second \( P_n \)-polygon \( \Phi_2 \) as follows: The boundary of \( \Phi_2 \) consists of the union of two polygonals the first of which is \( \text{Conv}(P_n) - e \) and the second is the polygonal defined by the subsequence \( S \) of \( u=p_1,...,p_n=v \). (See Figure 2(b).) It is now clear that the union of \( \Phi_1 \) and \( \Phi_2 \) covers \( \text{Conv}(P_n) \).

Proof of Theorem 1: Since the union of \( \Phi_1 \) and \( \Phi_2 \) covers \( \text{Conv}(P_n) \) and \( Q_m \) is contained in \( \text{Conv}(P_n) \) then either \( \Phi_1 \) or \( \Phi_2 \) contains at least half of the elements of \( Q_m \).

Next we prove that the bound in Theorem 1 is asymptotically tight. To prove this, consider a set \( P_n \) with \( n \) points such that \( \text{PConv}(P_n) \) is a triangle. Around each point \( p \in P_n \) in the interior of \( \text{Conv}(P_n) \) draw a circle \( C_p \) with radius \( \epsilon \), \( \epsilon \) small enough. Place \( r \) points uniformly distributed on \( C_p \), where \( r \) is large enough. For each vertex \( p_i \) of \( \text{PConv}(P_n) \), let \( \alpha_i \) be the internal angle of \( \text{PConv}(P_n) \) at \( p_i \); Place \( (\alpha_i/2\pi)r \) points uniformly at distance \( \epsilon \) from \( p_i \) within \( \alpha_i \) for \( i=1,2,3 \). Let \( Q \) be the set of points placed on the small circles around the points of \( P_n \). Clearly \( Q \) contains \( (n-3)r + r = (n-2)r \) points. Since the sum of the internal angles of any \( P_n \)-polygon is \((n-2)\pi\), then it encloses at most \([(n-2)r]/2 + n \) points. It follows that any such polygon contains at most \(|Q|/2 + n \) points. As \( r \to \infty \) this converges to \(|Q|/2 \).
It is natural to ask the following:

*Can we obtain some general conditions that guarantee that a point set $Q_m$ is enclosable by a given point set $P_n$? Is there a condition on $m$ which guarantees that $Q_m$ is enclosable by $P_n$?*

The answers to these questions seem to be linked to the size of Conv(P).

**Theorem 2:** If $Q_m$ is contained in the interior of Conv($P_n$) and $m (2 \log(m) + 1) \leq |\text{Conv}(P)|$ then $Q_m$ is always enclosable by $P$.

Here are some preliminary results needed to prove our result. The following lemma is given without a proof:

**Lemma 2:** Let $P_s$ be a point set and let $x$, $y$ and $z$ be three consecutive vertices of PConv($P_s$) and $q$ be a point in the interior of Conv($P_s$). Then there is a $P_s$-polygon $\Phi$ that encloses $q$ such that all edges of PConv($P_s$) are in $\Phi$, except possibly the edges joining $x$ to $y$ and $y$ to $z$ (See Figure 3.)

![Figure 3](image)

We recall the following well known result:

**Theorem 3 (Borsuk-Ulam):** Given any two collections of points $P$ and $Q$ on the plane, there is a line that simultaneously bisects them both.

**Sketch of the proof of Theorem 2:** Assume that $|\text{Conv}(P_n)| \geq m(2 \log(m) + 1)$ and that $m = 2^i$ is a power of 2. (Other forms for $m$ are handled similarly.) By theorem 3, we can simultaneously bisect the vertex set of PConv($P_n$) and $Q_m$ with a line $L$. Iterate this process $i$ times until $Q_m$ has been splitted into singletons and the vertices of
PConv(P_n) have been divided into 2^i subsets, each with size at least |Conv(P_n)| / 2^i that is into subsets of size at least 2\log(m)+1 (See Figure 4.) Clearly for each point q of Q_m we define in a "natural way" a polygon \Phi_q that contains it and whose vertices are:

a) either intersection points of the lines used to split Q_m, or
b) at least 2\log(m)+1 points of PConv(P_n) which constitutes one of the 2^i subsets of the vertices of PConv(P_n)

Since the splitting process is repeated i times, \Phi_q contains at most i=\log(m) edges which arise from the lines used to split Q_m (See figure 4(a).) It is easy now to verify that a) and b) together imply that in each \Phi_q there are three consecutive vertices of PConv(P_n) (See Figure 4(a).)

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure4a.png}
\includegraphics[width=0.4\textwidth]{figure4b.png}
\caption{Figure 4.}
\end{figure}

Now in each \Phi_q we can find a polygon enclosing q as in Lemma 2 (See Figure 4(b).) Finally joining all of the previously obtained polygons and deleting the lines used to split Q_m we obtain our desired P_n-polygon enclosing Q_m. This ends our proof.

References