# An Algorithm for Constructing the Convex Hull of a Set of Spheres in Dimension d*

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## Abstract

We present an algorithm which computes the convex hull of a set of $n$ spheres in dimension $d$ in time $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$. This algorithm is worst-case optimal in any dimensions. It can also be used to compute the convex hull of a set of $n$ homothetic objects of $E^d$. If $k$ is the combinatorial complexity of one of the objects, the worst-case time needed is $O(k(n^{\lceil \frac{d}{2} \rceil} + n \log n))$.


## 1 Introduction

Though the complexity and the computation of the convex hull of a set of points in any dimensions is a problem which has been studied extensively, only a few results about the convex hull of a set of spheres are known. The previous results, which are given below, are only for the case $d = 2$ and 3, and, as far as we know, there were no results about the computation of the convex hull of a set of homothetic objects .

The convex hull of a set of spheres is the smallest convex body that contains the spheres. In two dimensions, it consists of line segments and arcs of circles. In three dimensions, the convex hull is composed of three different kinds of facets (see Figure 1).

- Planar facets, which are triangles included in planes tangent to three spheres.
- Conical facets, which are parts of cones tangent to two spheres.
- Spherical facets, which are parts of the spheres of $E$.

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In the plane, the convex hull of a set of disks can be computed in $\Theta(n \log n)$ time (see [Rap92]). In three-dimensional space the complexity of the convex hull of a set of $n$ spheres is $\Theta(n^2)$ in the worst case, even for collections of pairwise disjoint spheres ([SS90]).

The paper is organized as follows: In the next section we give a lower bound on the complexity in any dimension. In section 3 we present the algorithm which computes the convex hull of a set of spheres and we show in section 4 by studying its complexity that it is time optimal in dimension 3 and in even dimensions. In section 5 we extend our results to homothetic convex objects.

In the sequel we will denote by $CH(\mathcal{O})$ the convex hull of a set of objects $\mathcal{O}$.

## 2 A Lower Bound

We want to show that the complexity of the convex hull of a set of $n$ spheres in dimension $d$ ($d \geq 3$) is $\Omega(n^{\lceil \frac{d}{2} \rceil})$.

By the upper bound theorem, the complexity of the convex hull of a set of $n$ points in dimension $d$ is $\Theta(n^{\lfloor \frac{d}{2} \rfloor})$ in the worst case. A point can be considered as a sphere of radius 0. Therefore, the complexity of the convex hull of a set of $n$ spheres is at least equal to the complexity of the convex hull of a set of points, thus is $\Omega(n^{\lfloor \frac{d}{2} \rfloor})$. If $d$ is even, $\lfloor \frac{d}{2} \rfloor = \lceil \frac{d}{2} \rceil$ and we have finished. If $d$ is odd, we construct a set of spheres of $E^d$ which convex hull has complexity $\Omega(n^{\lceil \frac{d}{2} \rceil})$ starting from a an example of a set of points in dimension $d-1$ which convex hull has maximal complexity.

If $d$ is odd, let $M$ be a set of $n$ points (considered as spheres of radius 0) on the $(d-1)^{th}$ order curve $(\cos t, \sin t, \cos 2t, \sin 2t, ..., \cos \frac{d-1}{2}t \sin \frac{d-1}{2}t)$ of $E^{d-1}$. The convex hull of $M$ is combinatorialy equivalent to a cyclic polytope (see[MS71]). The complexity of a cyclic polytope is maximal (i.e. $n^{\lfloor \frac{d-1}{2} \rfloor}$). More precisely, there are $\theta(n^{\lfloor \frac{d-1}{2} \rfloor})$ $(k-1)$-faces ($\frac{d-1}{2} \leq k \leq d-1$).

The points of the set $M$ are on a sphere centered on $O$ with radius is $\sqrt{\frac{d-1}{2}}$.

We add a vertex $P$ on the $Ox_d$ axis. There is a one-one

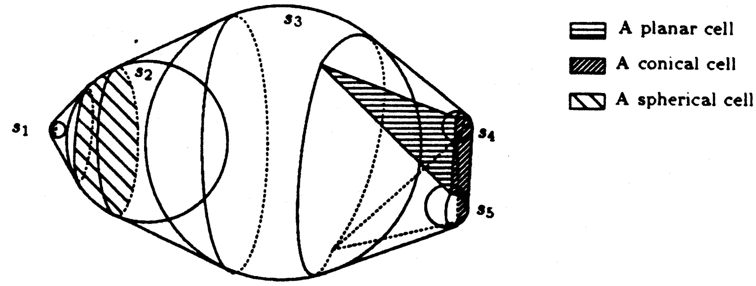
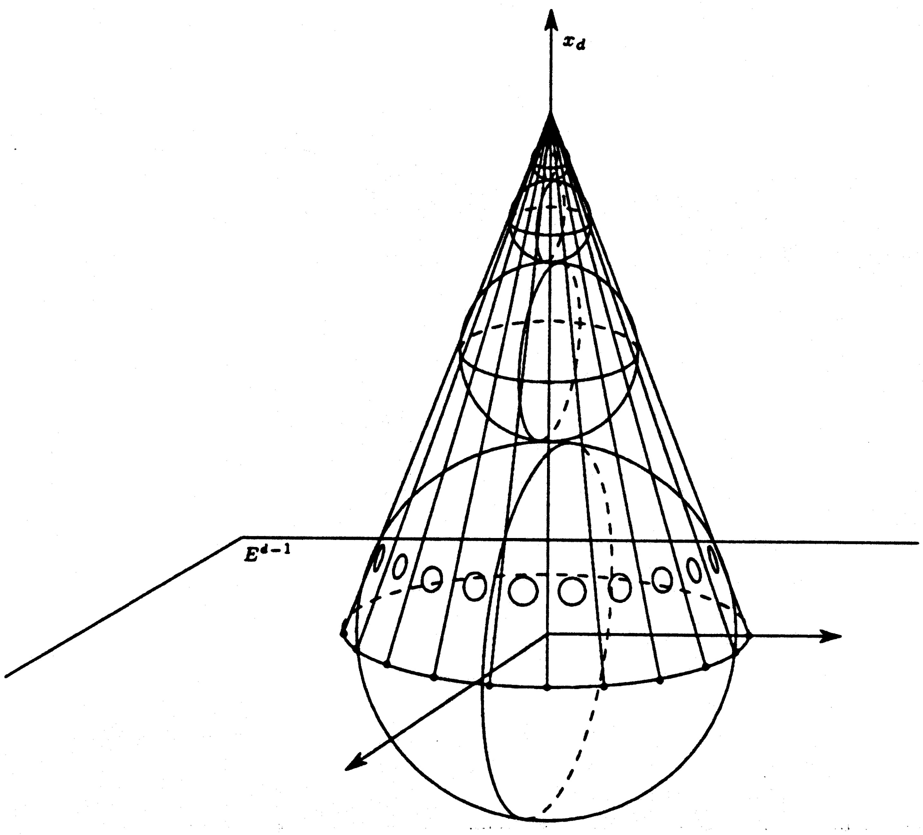



Figure 1: The convex hull of a set of spheres in 3 dimensions




• points on a $(d-1)^{th}$ order curve embedded in a sphere of $E^{d-1}$

Figure 2: A set of spheres whose convex hull has size $n^{\lceil \frac{d}{2} \rceil}$

correspondence between the $(k+1)$-faces of $CH(P \cup M)$ containing $P$ and the $k$-faces of $CH(M)$.

We now add a set $S$ of $n$ spheres to our set of $n+1$ points $P \cup M$ such that each sphere intersects $n^{\lfloor \frac{d-1}{2} \rfloor}$ $(d-2)$-faces of $CH(P \cup M)$. These spheres are centered on the $Ox_d$ axis and their radius decreases from $\sqrt{\frac{d-1}{2}}$ to 0. Each of the $n$ spheres is chosen such that there is just a small part outside $CH(P \cup M)$ which creates $n^{\lfloor \frac{d-1}{2} \rfloor}$ facets (see Figure 2).

In this way, the complexity of the convex hull of the $2n+1$ spheres $P \cup M \cup S$ is $n * n^{\lfloor \frac{d-1}{2} \rfloor} = n^{\lceil \frac{d}{2} \rceil}$.

**Theorem 1** *The complexity of the convex hull of a set of spheres in dimension $d$ is $\Omega(n^{\lceil \frac{d}{2} \rceil})$ in the worst-case.*

# 3 The Algorithm

We first introduce some notations, then recall some of the properties of duality, and finally give the algorithm that computes the convex hull of a set of spheres.

## 3.1 Notations

Let $A = \{S_1, ..., S_n\}$ be a set of spheres in $E^d$ of which we want to compute the convex hull.

We embbed $E^d$ in $E^{d+1}$ so that the hyperplane of $E^{d+1}$ : $x_{d+1} = 0$ contains all the spheres. The $(d+1)$-th axis will be called the vertical axis. Let $S$ be a sphere in $E^d$ with center $(x_1, ..., x_d)$ and radius $R$. Let $\phi$ be the mapping that associates to $S$ a point in $E^{d+1}$.

$$\phi : S \rightarrow \phi(S) = (x_1, ..., x_d, R)$$

Let $B$ be the set $\{\phi(S_1), ..., \phi(S_n)\}$ of $n$ points of $E^{d+1}$. Let $C_0$ be a half lower cone with arbitrary apex, axis $Ox_{d+1}$ and angle at the apex $\pi/4$.

For a sphere $S$ in $E^d$ let $\theta(S)$ be the half lower cone with apex $\phi(S)$ obtained by translation of $C_0$ . The intersection between the cone $\theta(S)$ and the hyperplane $x_{d+1} = 0$ is equal to the sphere $S$. Let $C$ be the set $C = \{\theta(S_1), ..., \theta(S_n)\}$ of $n$ cones of $E^{d+1}$ (see Figure 3). The intersection between the convex hull of the set of cones $C$ and the hyperplane $x_{d+1} = 0$ is equal to the convex hull of the set of spheres $A$.

Let $O'$ be a point inside $CH(B)$.

**Theorem 2** *Any hyperplane of $E^d$ supporting the convex hull of the set of spheres $CH(A)$ is the intersection with $x_{d+1} = 0$ of a unique hyperplane $H$ of $E^{d+1}$ which*

1. *supports the convex hull of the set of points $CH(B)$*
2. *is the translated of an hyperplane tangent to $C_0$.*
3. *is above $O'$ .*

*Conversely , the intersection of an hyperplane $H$, satisfying the above three properties, with the hyperplane $x_{d+1} = 0$ is an hyperplane of $E^d$ supporting the convex hull of the set of spheres $CH(A)$*

*Proof:* Each hyperplane of $E^d$ supporting the convex hull of the set of spheres $CH(A)$ is the intersection with $x_{d+1} = 0$ of a unique hyperplane $H$ which supports the convex hull of the set of cones $C$ and is tangent to at least one cone along a generatrix.

This means $H$ supports the convex hull of the set of points $CH(B)$ and is the translated of an hyperplane tangent to $C_0$.

As $H$ supports $CH(C)$, it is above $O'$.

Conversely, if $H$ supports $CH(B)$, it is either above or below $CH(B)$. So as $H$ is above $O'$, it is above $CH(B)$. As $H$ is also the translated of an hyperplane tangent to $C_0$, it supports the convex hull of the set of cones $CH_C$. Its intersection with $x_{d+1} = 0$ is an hyperplane of $E^d$ supporting $CH(A)$. □

## 3.2 Duality

We use duality to transform the conditions of the above theorem into simpler ones. Duality with respect to $O'$ is a one-one transformation which maps points of $E^d$ distinct from $O'$ to hyperplanes of $E^d$ which do not contain $O'$. Let $M_0$ be a point of $E^d$ distinct from $O'$. The hyperplane $H_0$ dual to $M_0$, is defined by the following relation:

$$H_0 = \{X \in E^d \mid M_0.X = 1\}$$

**In the Dual Space:**

- The dual of $CH(B)$ is a polytope $P$.
- The dual of the set of hyperplanes which are the translated of the hyperplanes tangent to $C_0$ is a cone $U$ with apex $O'$, axis $O'x_{d+1}$, and angle at the apex $\pi/4$.
- The dual of the set of hyperplanes above $O'$ (with respect to the $x_{d+1}$ axis) is the half space $x_{d+1} \geq 0$.

*Proof:*
First assertion: The dual of $CH(B)$ is a polytope $P$ limited by the hyperplanes dual to the vertices of $CH(B)$. Every $k$-face of $CH(B)$ corresponds to a unique $(d-k)$-face of $P$, for $0 \leq k \leq d$.

Second assertion: Let $H_2$ be an hyperplane tangent to $C_0$. The point dual to $H_2$ belongs to the line $L_2$ issued from $O'$ and normal to this hyperplane $H_2$. The dual of the set of all the hyperplanes parallel to $H_2$ is $L_2$. The angle of $H_2$ with the vertical axis is $\pi/4$. Therefore, $L_2$ has an angle $\pi/4$ with the vertical axis.

As $H_2$ moves around the cone $C_0$, staying tangent to it, $L_2$ moves on a cone $U$ with apex $O'$, axis $O'x_{d+1}$,

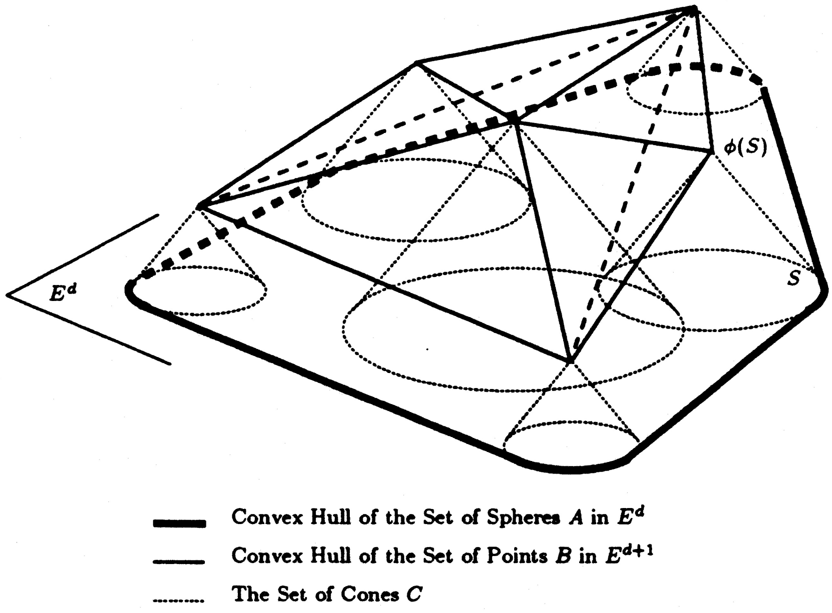



Figure 3: Embbeding the spheres in $E^{d+1}$

angle at the apex $\pi/4$. The dual of the set of all the hyperplanes tangent to $C_0$ and the translated of these hyperplanes is the cone $U$.

Third assertion: Let $H_3$ be an hyperplane which lies above $O'$. Let $M_3$ be the point dual to $H_3$. $M_3$ is on the half line issued from $O'$ and normal to $H_3$. $M_3$ is on the same side of $O'$ as $H_3$. Thus, the dual of the set of hyperplanes $H_3$ is the half space $x_{d+1} > 0$.
□

Thus, the dual of the set of hyperplanes supporting the convex hull of the set of points $CH(B)$, tangent to at least one cone of $C$ along a generatrix and above $O'$ is the set

$$I = P \cap U \cap \{x_{d+1} > 0\}$$

### 3.3 The Algorithm

Compute the convex hull $CH(B)$ and take a point $O'$ inside this convex hull.
Compute the simple polytope dual $P$ dual to $CH(B)$ with respect to $O'$.
Compute $I$, the intersection between $P$ and the cone $U$ and the half space $x_{d+1} \geq 0$.
Take the dual of $I$ (we are back to the initial space again) and intersect it with the hyperplane $x_{d+1} = 0$ to obtain the convex of the set of spheres $CH(A)$.

*Proof of correctness:*

The above theorem and the duality results yield: The intersection of the hyperplane of $E^{d+1}$ dual to a point of $I$ with the hyperplane $x_{d+1} = 0$ is an hyperplane of $E^d$ which supports the convex hull of the set of spheres $A$. The boundary of the intersection of all the half spaces limited by these supporting hyperplanes is the convex hull of the set of spheres $A$.

## 4 Complexity Analysis

Chazelle has shown that it is possible to compute the convex hull of a set of points in dimension $d$ in optimal time $\Theta(n^{\lfloor \frac{d}{2} \rfloor} + n \log n)$ (see [Cha91]). Thus, the worst-case time needed to compute the simple polytope dual to the convex hull of the set of points $A$ and to intersect it with the cone is

$$O(n^{\lfloor \frac{d+1}{2} \rfloor} + n \log n) = O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$$

This is optimal in any dimension. Simpler randomized algorithms can be found in [CS89, BDS+].

## 5 Extension to Homothetic Convex Objects

This algorithm generalizes to a set of homothetic convex objects having the same orientations. More precisely, let us take a convex object $CO$ of $E^d$ and let $A = \{CO_1, ..., CO_n\}$ be a set of convex objects, obtained from $CO$ by homothety and translation, of which we want to compute the convex hull . The main point is that the cone $U$ with angle at the apex $\pi/4$ is replaced by a more general cone $V$, which is no longer circular.

Let us associate a half lower cone $\theta(CO)$ of $E^{d+1}$ to $CO$ by taking an arbitrary apex $\phi(CO)$ above the object such that the vertical projection of the apex on $E^d$ is inside the convex object. $\theta(CO)$ is the half cone consisting of the half lines issued from $\phi(CO)$ and tangent to $CO$. Now we may associate a translated cone homothetic to $\theta(CO)$ to any object homothetic to $CO$. As before $B = \{\phi(CO_1), ..., \phi(CO_n)\}$ and $C = \{\theta(CO_1), ..., \theta(CO_n)\}$.

Arguments similar to those of section 3 can be used. If we replace the half lower cone $C_0$ defined in section 3 by $\theta(CO)$, Theorem 2 still holds. Condition 2 is then: The hyperplane $H$ is the translated of an hyperplane tangent to $CO$.

The dual of the set of hyperplanes $H$ satisfying condition 2 is a general cone $V$ with apex $O'$, which is no longer circular.

The algorithm of section 3 can be used if we replace the cone $U$ by $V$.

Let $k$ be the combinatorial complexity, i.e. the total number of (curved) faces, of the convex object $CO$. We assume that the dual of a curved face can be computed in $O(1)$ time so as its intersection with an hyperplane.

The complexity of the cone $V$ is also $k$ . To compute $I$ we have to intersect a polytope $P$ of complexity $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$ with $V$ and with the half space $x_{d+1} > 0$. The worst-case time needed to compute this intersection is $O(k(n^{\lceil \frac{d}{2} \rceil} + n \log n))$. Thus, the time needed to compute the convex hull of $n$ convex objects of complexity $k$ in dimension $d$ is $O(k(n^{\lceil \frac{d}{2} \rceil} + n \log n))$.

For example, if the convex objects are ellipsoids, we have $k = 1$. Thus, the time needed to compute the convex hull of $n$ homothetic ellipsoids in dimension $d$ is $O(n^{\lceil \frac{d}{2} \rceil} + n \log n)$.

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