Delaunay Triangulations and Computational Fluid Dynamics Meshes

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Abstract

In aerospace computational fluid dynamics (CFD) calculations, the Delaunay triangulation of suitable quadrilateral meshes can lead to unsuitable triangulated meshes. In this paper, we present case studies which illustrate the limitations of using structured grid generation methods which produce points in a curvilinear coordinate system for subsequent triangulations for CFD applications. We discuss conditions under which meshes of quadrilateral elements may not produce a Delaunay triangulation suitable for CFD calculations, particularly with regard to high aspect ratio, skewed quadrilateral elements.

1 Introduction

In computational fluid dynamics (CFD) applications, the problem domain must be discretized into meshes (or grids) over which the governing equations of fluid dynamics are solved. The two major classes of grids for aerospace CFD applications are structured grids and unstructured grids. Structured grids are curvilinear grids designed so that the neighbors to any element are implicitly known. These grids have been studied for quite some time and techniques for their construction are well understood [2]. Unstructured grids are composed of elements in which neighbors must be explicitly listed. The component elements are usually, but not necessarily, triangular. These grids are currently not as widely used in aerospace applications, and have been the object of recent interest [3, 4, 5]. Several properties of Delaunay triangulation [6, 7, 8, 9] make it attractive to use in unstructured grid generation. However, a major drawback to this method is the need for a separate method of point generation; a straightforward approach to this drawback is the use of a structured grid generator to create the necessary points [10, 11, 12, 13].

Certain features of structured grids are useful to maintain in unstructured grids. Most commonly, structured grids are body-fitted curvilinear meshes where contours follow the object boundaries. Structured grids may also contain very high aspect ratio elements. This allows properties of the problems being solved to be exploited; in general, very high solution gradients can exist perpendicular to surfaces, and very small solution gradients tangent to surfaces. Unstructured meshes should exhibit the same "structure" which orients the cells along a feature of interest. When a structured grid contains high aspect ratio grid cells, the Delaunay triangulation of the grid points may not maintain the original grid lines of the structured grid (Figures 1 and 2). Orientation of the cells in a particular direction is lost when this occurs.

This study investigates conditions under which Delaunay triangulations of points from structured grids will maintain the original grid structure. The effects of skew and aspect ratio on the Delaunay triangulation are studied.

2 Preliminaries

First consider a general case of points distributed arbitrarily along straight lines a distance $h$ apart, with the points no more than $s$ apart, and $s > h$. What relationship must exist between $s$ and $h$ so that the lines are guaranteed to be in the Delaunay triangulation of the points.

Let $P_1 = (-s/2, 0)$ and $P_2 = (s/2, 0)$ be two points on the line $y = 0$ (the results can be generalized to any location, but these points were chosen for simplicity of analysis). One scenario could have points on successive contours shifted by $s/2$, e.g., $R_1 = (0, h)$
and \( R_2 = (0, -h) \). For the edge \( P_1P_2 \) to be included in the triangulation of the points, the circumcircle for \( P_1, P_2, R_1 \) must not contain \( R_2 \), and likewise, the circumcircle for \( P_1, P_2, R_2 \) must not contain \( R_1 \). The circumcenter of \( P_1, P_2, R_1 \) is \((0, h/2 - s^2/8h)\). The circumcenter of \( P_1, P_2, R_2 \) is \((0, -h/2 + s^2/8h)\). To guarantee that edge \( P_1P_2 \) is in the triangulation, then

\[
\begin{align*}
-h/2 + s^2/8h &< h/2 - s^2/8h \\
\frac{s^2}{h^2} &< 4 \\
\frac{s}{h} &< 2
\end{align*}
\]

assuming distances positive.

Without knowing anything else about the distribution of points along the contour lines, we must conclude that the aspect ratio of the triangles can be no greater than 2 to guarantee that all original edges are in the triangulation. The introduction of structure into the points locations allows for larger aspect ratios under certain conditions. The remainder of this section defines some of the concepts to be used, and following sections explore this idea in further detail.

**Definition 1** A structured grid is an undirected graph \( G = (V, E) \) with vertex set \( V \) and edge set \( E \) such that

\[
V = \{ v_{i,j} \mid \forall v_{i,j} \exists (x_{i,j}, y_{i,j}) \in \mathbb{R}^2, \\
0 \leq i \leq m, 0 \leq j \leq n \}
\]

\[
E = \{ (u, w) \mid \forall v_{i,j} \in V, \\
i \neq m \Rightarrow (v_{i,j}, v_{i+1,j}) \in E \\
j \neq n \Rightarrow (v_{i,j}, v_{i,j+1}) \in E \}
\]

**Definition 2** A graph \( G = (V, E) \) is called Delaunay-embeddable if \( G \) is a subgraph of the Delaunay triangulation of \( V, D(V) \).

**Definition 3** A rectangular structured grid is a structured grid \( G = (V, E) \) where the location of each \( v_{i,j} \) is given by:

\[
\begin{align*}
x_{i,j} &= i \cdot s \\
y_{i,j} &= j \cdot h
\end{align*}
\]

where \( s, h \in \mathbb{R}, 0 \leq i \leq m, \) and \( 0 \leq j \leq n \).

Points in a rectangular structured grid have a constant spacing in \( x \) and a (possibly different) constant spacing in \( y \).

**Definition 4** The aspect ratio of a structured grid element is the ratio \( s/h \), where \( s \) is the length of a long side and \( h \) is the separation between the sides of length \( s \).

**Definition 5** A skewed structured grid with aspect ratio \( s/h \) is a structured grid where the location of each \( v_{i,j} \) is given by:

\[
\begin{align*}
x_{i,j} &= i \cdot s + j \cdot dz \\
y_{i,j} &= j \cdot h
\end{align*}
\]

where \( dz \leq s/2, 0 \leq i \leq m, \) and \( 0 \leq j \leq n \).

Points in a skewed structured grid still exhibit constant spacing in \( x \) and \( y \), however, at each level, a constant shift in \( x \) from the previous level results in a grid of parallelograms. The amount of skew in a skewed structured grid is \( \theta \), the angle formed by a perpendicular to one of the quadrilateral sides at a corner (Figure 3).

**Definition 6** The Delaunay angle cut-off is the angle beyond which a skewed structured grid is no longer Delaunay-embeddable.

**Definition 7** A simple stretched structured grid is a structured grid where the location of each \( v_{i,j} \) is given by:

\[
\begin{align*}
x_{i,j} &= i \cdot s + \sum_{k=0}^{j} dz(1+\epsilon)^k \\
y_{i,j} &= \begin{cases} 
0, & j = 0 \\
\frac{\sum_{k=0}^{j-1} h(1+\epsilon)^k}{\sum_{k=0}^{j} (1+\epsilon)^k}, & j = 1, 2, \ldots, n 
\end{cases}
\end{align*}
\]

where \( dz \leq s/2, 0 < \epsilon < s, 0 \leq i \leq m, \) and \( 0 \leq j \leq n \).

The simple stretched structured grid is one which exhibits a constant multiplicative growth in spacing between levels. This will be referred to as a stretched structured grid for the remainder of the paper.

The following facts are to be noted: The center of any circle which passes through \( P_1 = (-s/2, 0) \) and \( P_2 = (s/2, 0) \) will lie on the line \( z = 0 \) (in general, the center will lie on the line which is the perpendicular bisector to line segment \( P_1P_2 \)). Delaunay triangulations have the following properties: For any convex quadrilateral in the triangulation, the diagonal is selected such that the minimum angle is maximized; the diagonal selected is not necessarily the shortest diagonal. For the degenerate case of four co-circular points, one of two edges may be selected; in such cases, the edge which is a member of the edge set \( E \) is chosen.
3 Skewness and Aspect Ratio

Given a set of quadrilaterals, ideally the principle direction of the skewed triangles should follow the original boundaries of the quadrilaterals. However, if there is skew (i.e., the quadrilaterals are parallelograms rather than rectangles), then it is possible for the Delaunay triangulation to break the quadrilateral boundaries. This section describes the conditions under which this happens.

The case of a skewed structured grid with $s > h$ is studied (Figure 4). The goal is to produce a Delaunay triangulation of the points in the vertex set $V$ such that each triangle lies between the lines $y = jh$ and $y = (j + 1)h$. In other words, we will determine the restrictions on the grid such that the skewed structured grid is Delaunay-embeddable.

First let us consider any parallelogram with aspect ratio $s/h$, with $s > h$. The short diagonal creates two angles, $\alpha$ and $\beta$, with $\beta$ opposite the side of length $s$. As the parallelogram is skewed by $\theta$, an amount $\delta$ is added to two opposing corners and subtracted from the remaining two corners, causing a shift of $dx$ to the upper two corner points (Figure 5).

**Lemma 1** As $\theta$ increases, $\alpha$ and $\beta$ both increase.

**Proof:** Consider points $P_3$ and $P_4$ at the upper corners of a rectangle. Shift these points a distance $dx$ relative to $P_1$ and $P_2$. This is the same as adding $\delta$ to angle $P_1P_2P_4$ and angle $P_1P_2P_3$. Originally, the diagonal from $P_3$ to $P_2$ made angles $\alpha$ and $\beta$ with the sides $P_1P_2$ and $P_3P_4$. Since we started with a rectangle, $\alpha$ and $\beta$ are guaranteed to be less than 90 degrees. Since $s > h$, $|P_3P_2| > |P_1P_3|$. When the points $P_3$ and $P_4$ are shifted a distance $dx$ to new points $P'_3$ and $P'_4$, two new angles $\alpha'$ and $\beta'$ are achieved. Let $\theta_1 = \angle P_3P_2P'_3$ and $\theta_2 = \angle P_2P_4P'_4$. $\alpha' = \alpha + \theta_1$ and $\beta' = \beta - \theta_1 + \theta_2$. It is immediately apparent that $\alpha' > \alpha$ and $\beta' > \beta$, because $|P_3P_2| > |P_1P_3|$ implies $\theta_2 > \theta_1$ while $dx \leq s/2$. \( \square \)

Now consider adjacent quadrilaterals $P_1P_2P_3P_4$ and $P_2P_3P_4P_5$ within a skewed structured grid. A skewed structured grid is Delaunay-embeddable for the degenerate case $dx = 0$ since the corner points of any quadrilateral will be co-circular. For $P_1P_2P_3P_4$, either $P_3P_4$ or $P_2P_5$ could be chosen as diagonals. The following facts are to be noted:

- $dx = 0$ and $s > h \Rightarrow \beta < 90$ and $\angle P_1P_2P_3 > \angle P_3P_2P_4$.
- $\angle P_1P_2P_3 \equiv \angle P_4P_3P_2$
- As a quadrilateral $P_1P_2P_3P_4$ is skewed, edge $P_2P_3$ will be in the Delaunay triangulation of its points.
- $\angle P_1P_2P_3$ is the smallest angle in the Delaunay triangulation of $P_1P_2P_3P_4$.

**Lemma 2** If $\beta < 90$ degrees, then a skewed structured grid is Delaunay-embeddable.

**Proof:** From Lemma 1, we know that $\beta$ increases as $dx$ increases. It follows that $\beta$ approaches 90 degrees as $dx$ increases. Since the lines are a constant distance $h$ apart, when $\beta = 90$ degrees, the diagonal line $P_3P_2$ becomes a perpendicular bisector for the line $P_1P_0$, and at this point $P_3P_2$ also bisects the angle formed by $P_1P_2P_3$.

The convex quadrilateral formed by points $P_3P_2P_3P_4$ (Figure 6) has as its diagonal either $P_3P_4$ or $P_2P_3$. When $dx = 0$, we know that $P_3P_4$ is selected as the diagonal, because $\angle P_4P_3P_2$ is the larger of the smallest angles in the possible triangulations of this quadrilateral. As a skew of $\theta$ is introduced to the grid, $P_3$ moves faster than $P_3$, so $\angle P_3P_2P_5$ grows faster than $\angle P_1P_2P_3$ (from lemma 1 we know that both will increase). When $\beta = 90$ degrees, the two angles are equal, and because $s > h$, $\angle P_4P_3P_2 < \angle P_1P_2P_3$. When $\beta$ exceeds 90 degrees, then because $P_3P_2P_3$ is increasing faster, it becomes the larger of the smallest angles in the possible Delaunay triangulations of $P_3P_2P_3P_4$, and $P_3P_2$ is then selected as the diagonal. \( \square \)

When this occurs, the Delaunay triangulation no longer includes the original quadrilateral boundaries.

**Lemma 3** If $s/h \leq 2$, then a skewed structured grid is Delaunay-embeddable.

**Proof:** At $dx = 0$, $\beta < 90$ degrees. $\beta$ reaches a maximum at $dx = s/2$. When $s/h < 2$, $\beta < 90$ degrees at $dx = s/2$, and the grid remains Delaunay-embeddable. When $s/h = 2$, $\beta = 90$ degrees at $dx = s/2$, and the grid is Delaunay-embeddable. \( \square \)
Theorem 1 As aspect ratio increases, the Delaunay angle cut-off for a skewed structured grid decreases.

Proof: From Lemma 2, we know that the Delaunay angle cut-off occurs when $\beta = 90$ degrees. At this point, the Pythagorean theorem gives

$$dx = \frac{s - \sqrt{s^2 - 4h^2}}{2}$$

The Delaunay angle cut-off $\theta^*$ is defined by

$$\tan \theta^* = \frac{dx}{h} = \frac{s - \sqrt{s^2 - 4h^2}}{2h}$$

$$= \frac{1}{2} \left( \frac{s}{h} - \frac{\sqrt{s^2 - 4}}{\sqrt{h^2 - 4}} \right)$$

For $s/h \leq 2$, there is no Delaunay angle cut-off (lemma 3). As $s/h$ increases, $\theta^*$ decreases (Figure 7).

Therefore, whether or not a structured grid is Delaunay-embeddable is dependent on aspect ratio, and as aspect ratio increases, the "tolerance" for skew angles decreases. As $s/h$ gets larger, the value of the Delaunay angle cut-off $\theta^*$, where the Delaunay triangulation no longer includes the original quadrilateral boundaries, approaches 0.

For the case of a monotonic stretched structured grid, the skew angle derived above is a lower bound.

Theorem 2 As aspect ratio increases, the Delaunay cut-off angle of a stretched structured grid decreases.

Proof: We need only consider two adjacent quadrilaterals at a time from the stretched structured grid with aspect ratios $s/h$ and $s/(h+\epsilon)$. The skew angle $\theta$ can be derived from the circumcircles for the two interior triangles, and is defined by

$$\tan \theta = \frac{dx}{h} = \frac{s - \sqrt{s^2 - 4h^2 - 4ch^2 - \epsilon^2h^2}}{2(\epsilon + h)}$$

$$= \frac{1}{2 + \epsilon} \left( \frac{s}{h} - \sqrt{\frac{s^2 - 4}{h^2}} - \epsilon(4 + \epsilon) \right)$$

Since successive levels have different (increasing) values of $\theta$ for $\epsilon$ positive, the grid will be Delaunay-embeddable when $\theta$ is based on the largest aspect ratio elements found in the grid. $\Box$

4 Conclusions

We have shown the limitations of Delaunay triangulations of points from structured grids for aerospace applications. For the general case of points distributed along fixed contours, we have shown a restriction on the aspect ratio for which Delaunay triangulations can be directly obtained. By imposing a structure on the point distribution, we have demonstrated the relationship between aspect ratio and quadrilateral element skew on the maintenance of contours from structured grids.

References


Figure 3 – A Skewed Structured Grid Element

Figure 4 – A Skewed Structured Grid

Figure 5 – Effects of Increasing Skew

Figure 6 – $\beta < 90$ and $\beta > 90$

Figure 7 – Delaunay Angle Cut-Off vs. Aspect Ratio

$r = \tan\left(\frac{1}{2} \left( \theta - \sqrt{\theta^2 - 4} \right) \right)$