The Edge Guard Problem for Spiral Polygons

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Abstract

In this paper we have solved the edge guard problem for spiral polygons. We have showed that $\lfloor (n+2)/5 \rfloor$ edge guards are necessary and sufficient to cover a spiral polygon. It has been shown by Aggarwal [Theorem 4.2–O'Rourke 87] that $\lfloor (n+2)/5 \rfloor$ diagonal guards are necessary and sufficient to cover a spiral polygon. Edge guards are more restrictive than diagonal guards. Hence the previous theorem can be got as a corollary using our theorem. The necessary condition of the edge guard problem for spiral polygons has not been investigated although the diagonal guard problem for the same has been solved [Sec 4.3–O'Rourke 87]. The necessary proof of the edge guard problem follows from the necessary condition of the diagonal guard problem but we have given an alternate proof of necessity.

KEYWORDS : Computational Geometry, visibility, edge guards, spiral polygon.

INTRODUCTION

A point $x \in P$, a polygon, is said to cover a point $y \in P$, if the line segment xy is completely in the interior of P i.e., $xy \subseteq P$. A set of guards are said to cover a polygon P if these guards when positioned at some vertices of the polygon can cover the whole polygon (such guards are called vertex guards). A mobile guard is a guard who 'patrols' a fixed line segment which is completely contained within the polygon P. A point x is said to be covered by a line segment l, if there exists a point $y \in l$ such that $xy \subseteq P$. An *edge guard* is a mobile guard who 'patrols' an edge. A *diagonal guard* is a mobile guard who 'patrols' any internal diagonal between any two vertices of P. Thus an edge guard is also a diagonal guard but not vice versa.

A *reflex chain* of a polygon is a sequence of consecutive reflex vertices. A spiral polygon is a polygon with atmost one reflex chain. The necessary and sufficiency of the number of vertex guards required for covering a spiral polygon is $\lfloor n/3 \rfloor$. The necessary condition was established using a distorted "comb" example [O'Rourke 87]. Sufficiency is given by *Chavatal's* theorem ($\lfloor n/3 \rfloor$ guards are occasionally necessary and always sufficient to cover a polygon with n vertices [Chavtal 75])

The edge guard problem for spiral polygons has not been studied. In the following paper we show that there exist spiral polygons which require $\lfloor (n+2)/5 \rfloor$ edge guards to cover them completely

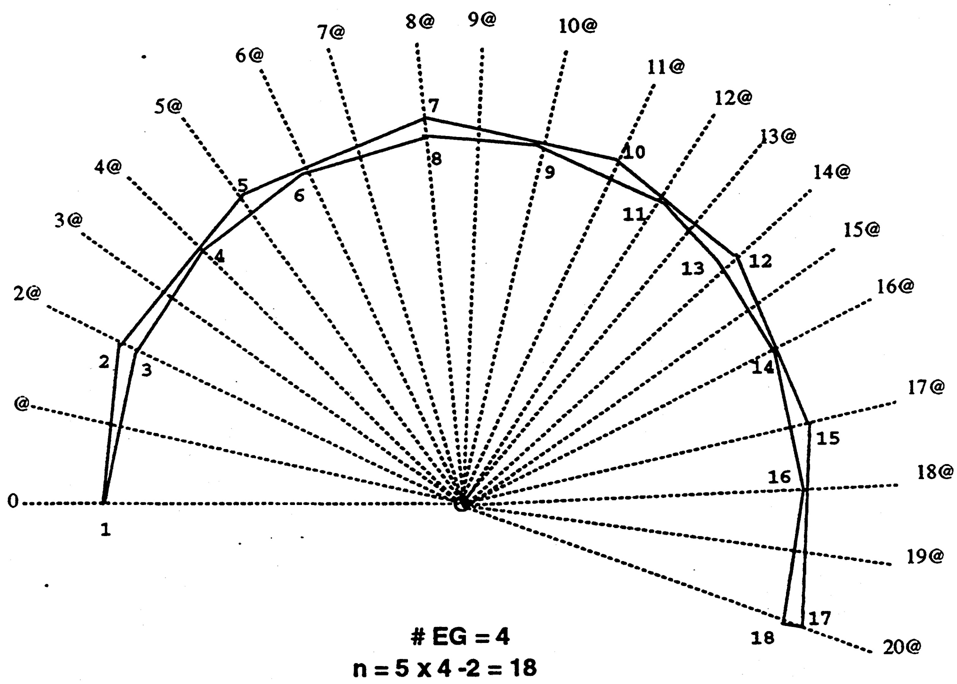

Figure 1:

and we prove that $\lfloor (n+2)/5 \rfloor$ edge guards are sufficient to cover any spiral polygon [1].

THEOREM

$\lfloor (n+2)/5 \rfloor$ edge guards are necessary and sufficient to cover any spiral polygon.

Proof of Necessity:

A generic example for $n = 5k-2$ consists of $3k-1$ equally spaced vertices on the circumference of a circle and $2k-1$ more equally spaced vertices on a slightly larger concentric circle. The consecutive vertices on the smaller circle taken in clockwise order form a reflex chain. The consecutive vertices on the larger circle forms another chain (non-reflex). The endpoints of these two chains are joined to give a spiral polygon, refer to Figure 1. The consecutive vertices of the reflex chain are positioned at angles $2i$@, i=0,1,2,.....,$3k-2$. The consecutive vertices of the non-reflex chain are positioned at angles $(3i+2)$@, i=0,1,2,.......,$2k-2$. The -ve x-axis is considered as the starting point and we move in the clockwise direction. The outer radius is chosen close enough to the inner radius so that the edges adjacent to all even vertices v on the non-reflex chain barely touches the vertices of the reflex chain that occur on either side of the vertex v.

LEMMA: The spiral polygon constructed above requires $\lfloor (n+2)/5 \rfloor$ edge guards.

PROOF: To cover vertex 1 we have to have an edge guard on (1,2),(1,3), (2,5) or (3,4). An edge guard on (2,5) covers everything covered by having an edge guard on any of the other three and something more-so we have to have an edge guard on (2,5) in order to minimize the number of edge guards required. An edge guard on (2,5) covers the sub-spiral polygon(SSP) (1,3,4,6,7,5,2), refer

[1] this fact has also been proved independently by Iliana Bjorling-Sachs [Iliana 93] in a different way

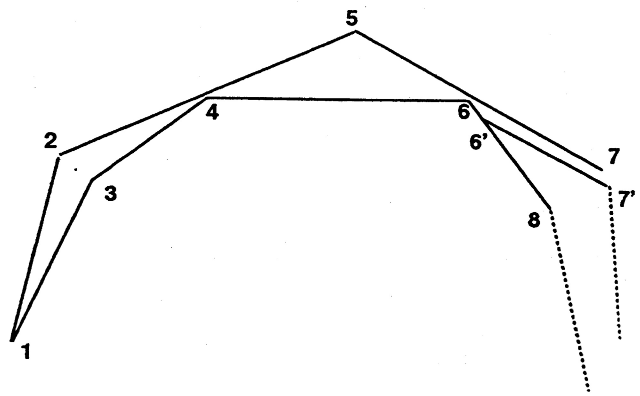

Figure 2:

to Figure 2. Consider the points $6'$ and $7'$ that are very close to points 6 and 7 on the otherside of the SSP. Join points $6'$ and $7'$ The polygon that remains after deleting the SSP and joining $6'$ and $7'$ is a spiral polygon similar to the original spiral polygon with $n' = n-5$ vertices. This process can be repeated till $n' < 3$ and in each step we remove 5 vertices. It will take $\lfloor (n+2)/5 \rfloor$ steps for n' to become less than 3 and in each step we use one edge guard. Therefore we require atleast $\lfloor (n+2)/5 \rfloor$ edge guards to cover the whole polygon •

For any n, ($(5k-2) < n < (5(k+1)-2)$, a spiral polygon is constructed as described above for $n' = (5(k+1)-2)$ and then vertices are deleted starting at the largest angle and proceeding in the anti-clockwise direction, ties (two vertices , one of the reflex chain and one of the non-reflex chain occurs at the same angle) are broken by choosing the vertex on the reflex chain as the vertex to be deleted first. $n'-n$ vertices are deleted and the endpoints of the reflex and non-reflex chains are joined to give the n-vertex spiral polygon. The necessity of $\lfloor (n+2)/5 \rfloor$ edge guards for this polygon follows from the necessity of $\lfloor (n+2)/5 \rfloor$ edge guards for the spiral polygon with $n'' = 5k-2$ vertices which is a part of this polygon. This is because :

$$\lfloor (n''+2)/5 \rfloor = \lfloor (n+2)/5 \rfloor$$

when $n'' = 5k-2$ and $(5k-2) < n < (5(k+1)-2)$

Proof of Sufficiency

Consider any spiral polygon. We begin by identifying a chain (SV) of atleast seven vertices, in the original spiral polygon. The original polygon can be considered to be consisting of SV and the remaining part of the original polygon after SV is removed. The SV is such that it is connected to the remaining part by two edges (incident upon two different vertices v and v' of SV), to form the original polygon. v and v' are connected and v and v' are duplicated and connected. Thus we get two spiral polygons S and S', where S consists of the SV with the last two vertices connected and S' consists of the remaining chain with the duplicated v and v' (which are connected). The cardinality of S is k and that of S' is $n-k+2$ (The 2 is because of the duplicated vertices), where $k \geq 7$.

We give a way of identifying SV and prove that S can be covered by just one edge guard.

IDENTIFICATION and PROOF : Let the vertices and the edges on the reflex chain be numbered $r_1, r_2, r_3, \ldots$ and $e_1, e_2, e_3, e_4, \ldots$ respectively in the order in which they occur in the clockwise

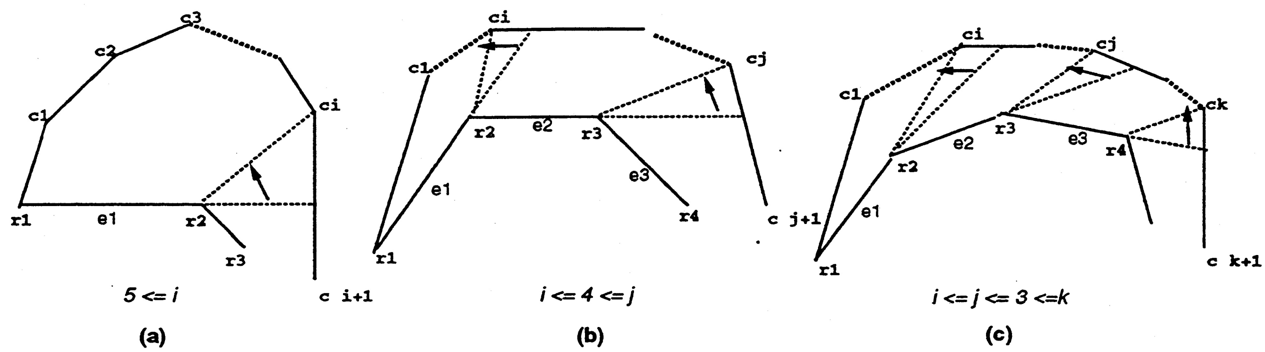

Figure 3:

direction and let the vertices on the convex chain be numbered $c_1, c_2, c_3, \ldots$..

• *Step* 1 :(Refer to Figure 3(a)) Extend e_1, to meet the convex chain at a point on an edge. Let this edge be incident upon the vertices c_i and $c_{(i+1)}$ (c_i occurs before $c_{(i+1)}$ in the convex chain) . Join r_2 to c_i. Consider the region above the cut. This is a convex region, because we have a convex chain and two other edges which meet at a convex angle(the two edges are the e_1 and the edge that we introduced between the r_2 and c_i), hence placing an edge guard on any edge will cover the polygon.. If this region has atleast 5 vertices from the convex chain, then SV is the vertices from the convex chain and the first two vertices of the reflex chain. v is c_i and v' is the r_2.

If we have less than 5 vertices from the convex chain than go to *Step* 2.

• *Step* 2 :(Refer to Figure 3(b)) Extend e_2, to meet the convex chain at a point on an edge. Let this edge be incident upon the vertices c_j and $c_{(j+1)}$ (c_j occurs before $c_{(j+1)}$ in the convex chain) . Join r_3 to c_j. We can decompose the resulting polygon into three convex polygons. The first one is got as in *Step* 1. For getting the 2nd convex polygon move along the convex chain in the clockwise direction starting at c_i , till we come to a vertex v such that the angle between c_i, r_2 and v is reflex (in case we do not encounter no such vertex and we reach c_j, then the second convex polygon is the polygon that remains of the original polygon after the 1st convex polygon is removed and the third polygon is the empty polygon). Join r_2 to the vertex that occurs just before v on the convex chain. This region is clearly convex. The region that remains is formed by a part of the original convex chain and three more edges (e_2 , the edge joining r_3 and c_j and the separation that got us the 2nd convex polygon (r_2, v)). This is also convex. All the three convex polygons have r_2 as a common vertex. Having a guard here(an edge guard patrols the whole edge– including the vertices) will cover the whole polygon. If the region that we got has atleast four vertices from the convex chain then SV consists of vertices from the convex chain and the first three vertices of the reflex chain. v is c_j and v' is r_3. If we have less than four vertices from the convex chain then we go to *Step* 3.

• *Step* 3 :(Refer to Figure 3(c)) The same procedure is repeated with e_3 . We will get c_k as the last vertex on the convex chain that is encountered . In this case we can decompose the polygon into 5 convex polygons. The first three (got as in *Step* 2) have r_2 as the common vertex and the other two have r_3 as a common vertex. Having a guard on vertex r_2 and r_3 will cover the whole polygon. Having a guard on e_2 will serve this purpose. If the number of vertices of the convex chain in the region got (v will be c_k and v' will be r_4)is atleast three then S will be the region that is got by joining c_k and r_4 and the chain containing $c_1, c_2, \ldots, c_k$ and r_1, r_2, r_3 and r_4.. Else go to *Step* 4.

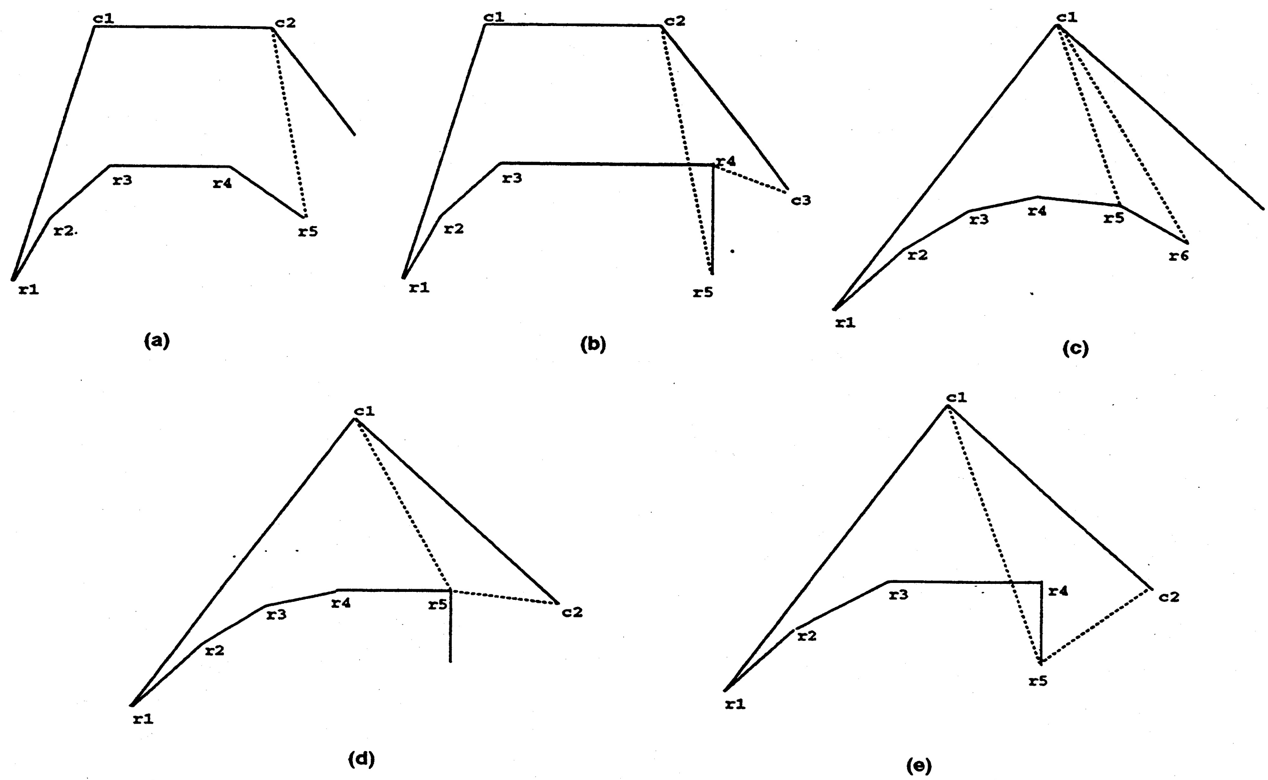

Figure 4:

• *Step* 4 : Now the number of vertices of the convex chain in the polygon got in *Step* 3 is less than three.

Case 1: The number of convex vertices is 2.

Consider the line (c_2, r_5), if this line is completely inside P, $(c_2, r_5) \subseteq P$ (Refer to Figure 4(a)), then v is c_2 and v' is r_5.

If $(c_2, r_5) \not\subseteq P$, then $(c_3, r_4) \subseteq P$,(Refer to Figure 4(b)) (this is because only one of the vertices of the reflex chain or one of the vertices of the convex chain can between these two vertices to break the line between them, in which case the reflexivity of the reflex chain or the convexity of the convex chain will be violated)[2]. In this case v is c_3 and v' is r_4. The convex polygons obtained in *Step* 3 have either c_1 or c_2 as one of the reflex vertices.(there is no way that all the vertices are r_i's –because of the reflexivity of the chain and a c_i, $i > 2$ cannot occur because of the above assumption). The new polygon got in this step is convex, because of the construction. Having a guard at c_2 will cover this polygon. Having an edge guard on the edge (c_1, c_2) will thus cover the whole polygon.

Case 2 : The number of convex vertices is 1.

Consider the line (c_1, r_5),

If $(c_1, r_5) \subseteq P$ (this line is completely inside P) then Consider the line (c_1, r_6), if this line is completely inside P, $(c_1, r_6) \subseteq P$ (Refer to Figure 4(c)) then
v is c_1 and v' is r_6.

If $(c_1, r_6) \not\subseteq P$ then $(c_2, r_5) \subseteq P$ [ARG].(Refer to Figure 4(d))
v is c_2 and v' is r_5.

[2]We will be using this argument in Case 2 below and will refer to it as [ARG]

If $(c_1, r_5) \not\subseteq P$ then $(c_2, r_4) \subseteq P$ [ARG].(Refer to Figure 4(e))

Consider the line (c_2, r_5), this line has to be completely inside P (because ,otherwise only r_4 could have come in between c_2 and r_5 and not any of c_i, $i > 2$ or r_j, $j > 5$ [ARG], in which case the region got in *step* 3 would have two vertices and is the same as *case* 1), $(c_2, r_5) \subseteq P$ then v is c_2 and v' is r_5.

If $(c_2, r_5) \not\subseteq P$ then $(c_3, r_4) \subseteq P$ [ARG].
v is c_3 and v' is r_4.

As in Case 1 , all the convex polygons have either c_1 or c_2 as one of their vertices. Therefore we can cover all the whole polygon by having an edge guard on (c_1, c_2).

Thus in all the cases we are able to cover the identified polygon with one edge guard and each of them has atleast seven vertices •

If we begin with a spiral polygon –each reduced polygon obtained by removing S (got as described above) is again going to be a spiral polygon because we just deleted a chain of vertices and then joined the end points hence there is no way of another reflex chain being introduced. Therefore the spiral nature of the polygon is preserved.

Thus by using up one edge guard for S we are left with the reduced polygon S', which has to be guarded. We repeat the same step of reducing the polygon till we are left with no more of the polygon to be guarded. In each step we remove 7 vertices from the polygon and add two vertices. Thus, we essentially remove 5 vertices in each step(except for the last step where we remove 7 vertices). Thus after $\lfloor (n-3)/5 \rfloor$ steps, we will be left with (atmost) a 7-vertex spiral polygon. Which can be guarded by one edge guard. This is true because we identified a seven vertex polygon each time and showed that this can be covered by one edge guard. If we just assume that this (last) 7-vertex polygon is actually a part of a larger polygon and is encountered in the beginning then we would have to delete this polygon to get a reduced polygon to go further. Essentially, we might encounter all type of polygons and we proved that we can guard any of them (having 7 vertices-this can be isomorphic to the sub-polygon that is left in the end) by one edge guard and delete that part and go ahead and cover the remaining polygon with one less guard. We use one guard for each step, thus we will need ($\lfloor (n-3)/5 \rfloor + 1$) i.e., $\lfloor (n+2)/5 \rfloor$ edge guards to cover the whole polygon. This proves the sufficiency condition.

Therefore $\lfloor (n+2)/5 \rfloor$ edge guards are necessary and sufficient to cover any spiral polygon. *QED* ••

Conclusion

In this paper we have proved that $\lfloor (n+2)/5 \rfloor$ edge guards are necessary and sufficient to cover any spiral polygon.

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