Acyclic Hierarchical Cell Complexes

Michela Bertolotto (*) , Elisabetta Bruzzone (**) , Leila De Floriani (*)

(*) Department of Computer and Information Sciences
University of Genova
Viale Benedetto XV, 3 - 16132 Genova (Italy)

(**) Elsag Bailey S.p.A.
Research and Development
Via Puccini, 2 - 16154 Genova (Italy)

Abstract

A hierarchical cell complex in $\mathbb{E}^n$ is a hierarchy of $n$-dimensional complexes such that each $n$-complex is the refinement of a cell belonging to its parent. In this paper, we define both hierarchical cell complexes and hierarchical simplicial complexes. Hierarchical simplicial complexes are used as a basis for multiresolution hypersurface modeling. We show that hierarchical complexes, in which each component is an acyclic complex, can be ordered back-to-front with respect to a viewpoint.

1 Introduction

A hierarchical cell complex $\mathcal{H}$ is a hierarchy of $n$-dimensional cell complexes in $\mathbb{E}^n$ such that each $n$-complex composing it is the refinement of a cell belonging to its parent complex in the hierarchy. Thus, a hierarchical complex can be viewed as a description of a cell complex at different degrees of abstraction. A special class of hierarchical cell complexes is provided by hierarchical simplicial complexes, which are basically hierarchies of simplicial complexes in $\mathbb{E}^n$. The motivation which has led us to the definition of hierarchies of complexes is the development of multiresolution representations of hypersurfaces describing multidimensional scalar fields for specific applications in volume visualization. When the data are available at irregularly spaced points, simplicial meshes connecting the data points are used as basic connection structure [Mes91,Wil92].

In volume visualization or spatial data analysis, large quantities of data are generally available. Thus, it is desirable to have a multiresolution model of a hypersurface which acts as a data compression mechanism on the data. To this aim, we have defined a hierarchical model of a hypersurface (i.e., a surface described by a function $z = f(x_1, \ldots, x_n)$) as a hierarchy of hypersurface models based on simplicial complexes [Ber93]. This model extends a hierarchical representation for surfaces defined in [Def92] as a digital elevation model. Hierarchical surface models have been developed in the literature for describing terrains at different levels of resolution (see [Def92]). No such models have been defined for three or more dimensions.

For visualization purposes (in particular direct volume rendering based on projective methods), it is fundamental that the cells of the complex used as connecting mesh for the data points can be ordered with respect to any given viewpoint. A visibility order of a set of objects from some viewpoint is a sequence such that, if object $A$ obstructs object $B$, then $B$ proceeds $A$ in the sequence. Such order exists if the obstruction relation, usually called $infront/behind$ relation, is acyclic with respect to the viewpoint. Edelsbrunner proved [Ede90] that the $infront/behind$ relation is acyclic for all cell complexes in $\mathbb{E}^n$ that can be obtained by projecting the boundary complex of a convex polytope in $\mathbb{E}^{n+1}$. The Delaunay triangulation of any finite point set, for instance, belongs to such class of complexes.

In the paper, we prove that the $infront/behind$ relation on any cell complex, obtained by partially or fully expanding a hierarchical cell complex, is acyclic, provided that each complex forming the hierarchical cell complex be acyclic. In particular, the expansion of Delaunay-based hierarchical simplicial complexes produces acyclic cell complexes.

2 Preliminaries

This Section introduces some definitions and notations.

Informally, a cell complex is a subdivision of an $n$-
dimensional topological space into cells. Here, we consider cell complexes in the \( n \)-dimensional Euclidean space \( \mathbb{E}^n \).

A \( k \)-cell in \( \mathbb{E}^n \), with \( 0 \leq k \leq n \), is a closed connected subset of \( \mathbb{E}^n \), such that its interior is homeomorphic to an open \( k \)-dimensional disk and whose boundary is not null. We denote with \( b(\gamma) \) and \( i(\gamma) \) the boundary and the interior, respectively, of a \( k \)-cell \( \gamma \). A \( k \)-cell \( \gamma \) is said to be a cell of order \( k \). A \( q \)-cell \( \gamma' \) is said to be a \( q \)-face (or, simply, a face) of a cell \( \gamma \) if and only if \( \gamma' \subseteq \gamma \). A face \( \gamma' \) of a cell \( \gamma \) is called a proper face of \( \gamma \) if and only if \( \gamma' \subset \gamma \). The set \( \{ \gamma' \mid \gamma' \text{ s-face of } \gamma, s \leq k \} \) is called the \( s \)-skeleton of \( \gamma \), and denoted with \( \gamma^{(s)} \).

A cell complex \( \Gamma \) of order \( n \) in \( \mathbb{E}^n \) (also called an \( n \)-complex) is a finite collection of cells satisfying the following properties:

(i) for each pair \( \gamma_1, \gamma_2 \in \Gamma \), where \( \gamma_1 \) is a \( p \)-cell, with \( p \leq n \), and \( \gamma_2 \) an \( m \)-cell, with \( m \leq n \),
\[ i(\gamma_1) \cap i(\gamma_2) = \emptyset; \]
(ii) the boundary \( b(\gamma) \) of a cell \( \gamma \in \Gamma \) is the union of cells belonging to \( \Gamma \);
(iii) for each pair \( \gamma_1, \gamma_2 \in \Gamma \) such that \( \gamma_1 \cap \gamma_2 \neq \emptyset \), \( \gamma_1 \cap \gamma_2 \) is the union of cells of \( \Gamma \).

Let \( \Gamma \) be an \( n \)-complex. The set \( \{ \gamma' \mid \gamma' \text{ s-cell of } \Gamma \} \), with \( 0 \leq s \leq n \), is called the \( s \)-skeleton of \( \Gamma \) and denoted with \( \Gamma^{(s)} \). Thus, an \( n \)-complex \( \Gamma \) is described by the \( (n+1) \)-tuple \( \Gamma = (\Gamma^{(0)}, \ldots, \Gamma^{(n)}) \) of its \( s \)-skeletons, with \( s = 0, \ldots, n \). The union of all the \( s \)-cells of \( \Gamma \), \( 0 \leq s \leq n \), regarded as point sets, is called the \textit{domain} of \( \Gamma \) and denoted \( D(\Gamma) \).

Let \( \Gamma_i \) and \( \Gamma_j \) be two \( n \)-complexes and let \( \gamma_1 \in \Gamma_i^{(n)} \) and \( \gamma_2 \in \Gamma_j^{(n)} \). Cells \( \gamma_1 \) and \( \gamma_2 \) are said to be \( s \)-adjacent, with \( 1 \leq s \leq n - 1 \), if and only if there exists an \( s \)-cell \( \gamma_p \) such that \( \gamma_1 \cap \gamma_2 \equiv \gamma_p \) and \( \gamma_p \) is union of proper faces of \( \gamma_1 \) or of proper faces of \( \gamma_2 \) (in this case, \( \gamma_1 \) and \( \gamma_2 \) are said to be \textit{adjacent along} \( \gamma_p \)). If \( V_1 \) and \( V_2 \) are the sets of vertices of \( \gamma_1 \) and \( \gamma_2 \), respectively, then \( \gamma_1 \) and \( \gamma_2 \) are said to be \textit{matching} along an \( s \)-cell \( \gamma_p \) if and only if \( \gamma_1 \) and \( \gamma_2 \) are \( s \)-adjacent along \( \gamma_p \) and \( V_1 \cap \gamma_p \equiv V_2 \cap \gamma_p \), where \( \gamma_p \) is the union of proper faces of both \( \gamma_1 \) and \( \gamma_2 \).

Let \( \Gamma_i \) and \( \Gamma_j \) be two \( n \)-complexes and \( \gamma \equiv D(\Gamma_i) \cap D(\Gamma_j) \) be an \( s \)-cell, with \( 1 \leq s \leq n - 1 \). \( \Gamma_i \) and \( \Gamma_j \) are said to be \( s \)-adjacent along \( \gamma \) if and only if \( \gamma \) is union of \( s \)-cells, each of which is the intersection of an \( s \)-cell of \( \Gamma_i \) and of an \( s \)-cell of \( \Gamma_j \). Moreover, if \( \Gamma_i \) and \( \Gamma_j \) are \( s \)-adjacent along \( \gamma \), then \( \Gamma_i \) and \( \Gamma_j \) are said to be \textit{matching} if and only if \( \gamma \cap V_i \subseteq \gamma \cap V_j \) or \( \gamma \cap V_j \subseteq \gamma \cap V_i \) and, for each pair \( \gamma_1, \gamma_2 \) with \( \gamma_1 \in \Gamma_i^{(n)} \) and \( \gamma_2 \in \Gamma_j^{(n)} \), such that \( \gamma_1 \cap \gamma_2 \equiv \gamma' \) (\( \gamma' \subseteq \gamma \)), \( \gamma' \) is union of proper faces of \( \gamma_1 \) or of proper faces of \( \gamma_2 \). This latter condition guarantees

![Figure 1: a) An example of 1-adjacent 2-complexes. b) An example of matching 2-complexes.](image-url)

that the intersection of two \( n \)-cells belonging to different matching complexes is always union of cells which are all proper faces of one of the two \( n \)-cells. Figure 1 shows an example of 1-adjacent 2-complexes and of matching 2-complexes.

An \( n \)-complex \( \Gamma \) is called \textit{convex} if and only if \( D(\Gamma) \) is a convex set and also the cells of \( \Gamma \) are convex sets. Two cells \( \gamma_1 \) and \( \gamma_2 \) of an \( n \)-complex \( \Gamma \) are said to be \( s \)-adjacent, with \( s < n \), if and only if there exists an \( s \)-cell \( \gamma \) which is a proper face of both \( \gamma_1 \) and \( \gamma_2 \). Note that \( \gamma_1 \) and \( \gamma_2 \) are matching along \( s \)-cell \( \gamma \). Two cells \( \gamma_1 \) and \( \gamma_2 \) are said to be \( s \)-connected if and only if there exists a sequence \( \{ \gamma_1, \ldots, \gamma_k \} \) of \( s \)-cells of \( \Gamma \) such that \( \gamma_i \equiv \gamma_{i-1} \equiv \gamma_2 \) and, \( \forall i = 1, \ldots, k - 1 \), \( \gamma_i \) and \( \gamma_{i+1} \) are \( s \)-adjacent. An \( n \)-complex \( \Gamma \) is said to be \textit{connected}, with \( 1 \leq s \leq n - 1 \), if and only if, for each partition of \( \Gamma \) in two subsets \( \Gamma_i \) and \( \Gamma_j \), there exist \( \gamma_1 \in \Gamma_i^{(n)} \) and \( \gamma_2 \in \Gamma_j^{(n)} \) such that \( \gamma_1 \) and \( \gamma_2 \) are \( s \)-connected. In the following, we will always consider convex, \((n - 1)\)-connected \( n \)-complexes, with polyhedral cells.

All definitions given above for cell complexes can be specialized for simplicial complexes. A simplicial complex of order \( n \) (also called an \( n \)-simplicial complex) in \( \mathbb{E}^n \) is a collection \( \Sigma \) of simplices satisfying the following properties:

(i) for each pair \( \sigma_1, \sigma_2 \in \Sigma \), where \( \sigma_1 \) is a \( p \)-simplex, with \( p \leq n \), and \( \sigma_2 \) an \( m \)-simplex, with \( m \leq n \),
\[ i(\sigma_1) \cap i(\sigma_2) = \emptyset; \]
(ii) the boundary \( b(\sigma) \) of a simplex \( \sigma \in \Sigma \) is the union of simplices belonging to \( \Sigma \);
(iii) for each pair \( \sigma_1, \sigma_2 \in \Sigma \) such that \( \sigma_1 \cap \sigma_2 \neq \emptyset \), \( \sigma_1 \cap \sigma_2 \) is a simplex of \( \Sigma \).
The class of simplicial complexes can be enlarged by extending the definition of a $k$-simplex to that of a generalized $k$-simplex. A generalized $k$-simplex can be either a $k$-simplex, if it has exactly $k + 1$ vertices, or a $k$-cell, with $m > k + 1$ vertices, in which exactly $k + 1$ vertices form a $k$-simplex $\sigma$ and the other $m - k - 1$ vertices lie on the boundary of $\sigma$. A generalized simplicial complex $\Sigma$ of order $n$ in $\mathbb{E}^n$ is a cell complex of order $n$ such that every cell is a generalized simplex.

Let $\Gamma$ be an $n$-complex, and $\gamma_1, \gamma_2 \in \Gamma^{(n)}$ such that $\gamma_1 \neq \gamma_2$. $\gamma_2$ obstructs $\gamma_1$ with respect to a given viewpoint $v_p$ in $\mathbb{E}^n$ if and only if there exists a ray $r$ emanating from $v_p$ such that $r \cap i(\gamma_1) \neq \emptyset$ and $r \cap i(\gamma_2) \neq \emptyset$ and, $\forall \, p \in r \cap i(\gamma_2)$ and $\forall \, p' \in r \cap i(\gamma_1)$, $p$ lies between $v_p$ and $p'$.

Let $\Gamma$ be an $n$-complex, and $\gamma_1, \gamma_2 \in \Gamma^{(n)}$ such that $\gamma_1 \neq \gamma_2$. If $\gamma_1$ and $\gamma_2$ are $(n - 1)$-adjacent, an order between the two cells with respect to point $v_p$ is defined as follows: $\gamma_1 <_{v_p} \gamma_2$ if and only if $\gamma_2$ obstructs $\gamma_1$. Such relation is called $\text{infront}/\text{behind}$ relation. The transition closure of $<_{v_p}$ is denoted with $<_{v_p}^*$.

A cycle in $\Gamma$ with respect to $v_p$ is a sequence of $n$-cells $[\gamma_1, \gamma_2, \ldots, \gamma_k]$ such that $\gamma_1 <_{v_p} \gamma_2 <_{v_p} \ldots <_{v_p} \gamma_k <_{v_p} \gamma_1$. The $\text{infront}/\text{behind}$ relation on $\Gamma$ is acyclic if and only if, for every viewpoint, no cycle exists. In this case, $\Gamma$ is said to be an $\text{acyclic complex}$. A visibility order of an acyclic $n$-complex $\Gamma$ with respect to a viewpoint $v_p$ is a linear sequence of the $n$-cells of $\Gamma$ such that if the cell $\gamma_2$ obstructs the cell $\gamma_1$, then $\gamma_1$ precedes $\gamma_2$ in the sequence. Any total order, that is consistent with the order defined by the $\text{infront}/\text{behind}$ relation on $\Gamma$, is a visibility order of the cells of $\Gamma$ relative to $v_p$ (i.e., for any pair of cells $\gamma_1$ and $\gamma_2$, if $\gamma_2$ obstructs $\gamma_1$, then $\gamma_1 <_{v_p}^* \gamma_2$).

If a viewpoint $v_p$ does not lie on any hyperplane containing an $(n - 1)$-face of a complex $\Gamma$, then it is possible to relate any pair of $n$-cells through relation $<_{v_p}$. If $v_p$ lies on such hyperplane, then it can be infinitesimally perturbed in order to generate rays which intersect the interiors of all the cells of $\Gamma$. An alternative way to solve such degeneracies is to establish an arbitrary order between the pairs of $n$-cells for which the $\text{infront}/\text{behind}$ relation cannot be stated.

3 Hierarchical Cell Complexes

In this Section, we introduce hierarchical cell complexes as hierarchies of cell complexes in $\mathbb{E}^n$, and the concepts of fully and partially expanded cell complex associated with such complexes.

Let $\mathcal{N} = \{\Gamma_0, \ldots, \Gamma_h\}$ be a family of $n$-complexes such that, $\forall \, i = 0, \ldots, h, \exists \, \gamma_i \in u_j^{(n)} \Gamma_j^{(n)}$, with $\gamma_i \equiv D(\Gamma_i)$. A hierarchy of cell complexes of order $n$ is a family $\mathcal{N}$ of $n$-complexes described by a tree $\mathcal{H} = (\mathcal{N}, \mathcal{E})$, where $\mathcal{E} = \{(\Gamma_i, \Gamma_j) \, | \, \Gamma_i, \Gamma_j \in \mathcal{N} \text{ and } \exists \, ! \, \gamma_j \in \Gamma_i^{(n)} \text{ such that } \gamma_j \equiv D(\Gamma_j) \text{ and } \Gamma_j^{(n)} \neq \{\gamma_j\}\}$. Each arc $(\Gamma_i, \Gamma_j) \in \mathcal{E}$ is labeled with $\gamma_j$.

Let $\mathcal{H} = (\mathcal{N}, \mathcal{E})$ be a hierarchy of cell complexes of order $n$. Any $\Gamma_i \in \mathcal{N}$ is called a component of $\mathcal{H}$. Let $\Gamma_i$ and $\Gamma_j$ be two components of $\mathcal{H}$. $\Gamma_i$ is said to be a child of $\Gamma_i$ and $\Gamma_i$ is said to be the parent of $\Gamma_j$ if and only if $(\Gamma_i, \Gamma_j) \in \mathcal{E}$. Similarly, the descendents of $\Gamma_i$ are all those complexes $\Gamma_k$ which are linked to $\Gamma_i$ by a path in $\mathcal{H}$ from $\Gamma_i$ to $\Gamma_k$, while the ancestors of $\Gamma_j$ are all the complexes $\Gamma_p$ which are linked to $\Gamma_j$ by a path from $\Gamma_p$ to $\Gamma_j$. If $\gamma_j \in \Gamma_i^{(n)}$ and $\Gamma_j \neq \Gamma_0$, then $\Gamma_j$ is said to be the direct expansion of $\gamma_j$ (and $\gamma_j$ the direct abstraction of $\Gamma_j$) if and only if $D(\Gamma_j) \equiv \gamma_j$ and $\Gamma_j$ is a child of $\Gamma_i$. In this case, cell $\gamma_j$ is called a macrocell. All complexes $\Gamma_k$ linked to $\Gamma_i$ by a path $[\Gamma_i, \Gamma_k_1, \ldots, \Gamma_k_m, \Gamma_k]$ in $\mathcal{H}$, where $\Gamma_k_1$ is the direct expansion of a macrocell $\gamma_j \in \Gamma_i^{(n)}$, are said to be expansions of $\gamma_j$, while all $n$-cells of such complexes are said to be descendant cells of $\gamma_j$.

A Hierarchical Cell Complex (HCC) of order $n$ is a hierarchy of cell complexes $\mathcal{H} = (\mathcal{N}, \mathcal{E})$ of order $n$, in which any two complexes $\Gamma_i$ and $\Gamma_j$ of $\mathcal{H}$ adjacent along an $s$-cell $\gamma$ $(1 \leq s < n)$ are matching along $\gamma$. Figure 2 shows an example of a hierarchical cell complex of order 2.

Let $\mathcal{H} = (\mathcal{N}, \mathcal{E})$ be an HCC. Let $\mathcal{N}'$ be a proper subset of
$N$ and $E'$ be the set of arcs of $E$ having their endnodes in $N'$. If $H' = (N', E')$ is a tree, then $H'$ is called a hierarchical subcomplex of $H$.

An HCC $H$ defines a hierarchical representation of a cell complex $\Gamma_H$, called the expanded complex associated with $H$. Such complex is defined on the basis of the process of cell refinement. In more details, the process of replacing a macrocell $\gamma_j$ in an n-complex $\Gamma_i$ with an n-complex $\Gamma_j$ such that $D(\Gamma_j) \equiv \gamma_j$ is called refinement. The complex $\Gamma_{ij}$ obtained by refinement of $\gamma_j$ with $\Gamma_j$ in $\Gamma_i$ is defined as follows:

$$\Gamma_{ij}^{(s)} = \Gamma_i^{(s)} \setminus \gamma_j^{(s)} \cup \Gamma_j^{(s)}, \quad (0 \leq s \leq n),$$

where $\gamma_j^{(s)}$ is the s-skeleton of $\gamma_j$ and $\Gamma_i^{(s)}$, $\Gamma_i^{(s)}$ and $\Gamma_j^{(s)}$ are the s-skeletons of $\Gamma_{ij}$, $\Gamma_i$ and $\Gamma_j$, respectively. We can, thus, define a function $\rho$ such that $\rho(\Gamma_i, \Gamma_j, \gamma_j) = \Gamma_{ij}$.

Thus, the expanded complex $\Gamma_H$ associated with $H$ is the cell complex obtained by iteratively refining all the macrocells belonging to the complexes in the hierarchy. More formally, we introduce the function $\rho_T$, such that $\rho_T(H) = \Gamma_H$, defined as follows:

$$\rho_T(H) = \begin{cases} 
\Gamma_0 & \text{if } H = \{\{\Gamma_0\}, \emptyset\} \\
\rho(\ldots(\rho(\Gamma_0, \rho_T(H_{k_1}), \gamma_{k_1}), \ldots, \rho_T(H_{k_p}), \gamma_p), \\
\text{with } \gamma_r \equiv D(\Gamma_{k_r}), \ r = 1, \ldots, p \\
\rho_T(H) & \text{otherwise}
\end{cases}$$

where $\Gamma_0$ is the root of $H$, $\Gamma_{k_1}, \ldots, \Gamma_{k_p}$ are all the children of $\Gamma_0$ and $H_{k_1}, \ldots, H_{k_p}$ are the subtree of $H$ rooted at $\Gamma_{k_r}$. It has been shown that both function $\rho$ and function $\rho_T$ define a cell complex [Ber93]. Thus, the cell complex $\Gamma_H$ defined by function $\rho_T$ applied to an HCC $H$ is said to be the (fully) expanded complex associated with $H$.

As we have already seen, an HCC $H$, rooted at a complex $\Gamma_0$, provides a hierarchical description of the refinements of the cells of $\Gamma_0$. Sometimes, only some of these refinements are of interest. Thus, we call partially expanded complex associated with $H$ any (fully) expanded complex associated with a hierarchical subcomplex of $H$. On the basis of this definition, if $H'$ is a hierarchical subcomplex of $H$, then the partially expanded complex associated with $H'$, obtained by full expansion of $H'$, is denoted with $\Gamma_{H'}$.

4 Hierarchical Simplicial Complexes

In this Section, we consider hierarchies of complexes whose components are simplicial complexes. We define two different types of hierarchical complexes, that we term generalized hierarchical simplicial complexes and hierarchical simplicial complexes, based on two different definitions of matching, called weak and strong matching, respectively.

Let $\Sigma_i = (\Sigma_i^{(0)}, \ldots, \Sigma_i^{(n)})$ and $\Sigma_j = (\Sigma_j^{(0)}, \ldots, \Sigma_j^{(n)})$ be two n-simplicial complexes and $V_i$ and $V_j$ their 0-skeletons (i.e., $V_i=\Sigma_i^{(0)}$ and $V_j=\Sigma_j^{(0)}$). If $\Sigma_i$ and $\Sigma_j$ are s-adjacent along an s-simplex $\sigma$, then $\Sigma_i$ and $\Sigma_j$ are said to be weakly matching along $\sigma$ if and only if

- $\sigma \cap V_i \subseteq \sigma \cap V_j$ or $\sigma \cap V_j \subseteq \sigma \cap V_i$;

- For each pair $\sigma_i, \sigma_j$ with $\sigma_i \in \Sigma_i^{(n)}$ and $\sigma_j \in \Sigma_j^{(n)}$, such that $\sigma_i \cap \sigma_j \equiv \sigma' \ (\sigma' \subseteq \sigma)$, $\sigma'$ is a proper face of $\sigma_i$ or $\sigma_j$.

Two n-simplicial complexes $\Sigma_i$ and $\Sigma_j$ are coincident, i.e., $\Sigma_i \equiv \Sigma_j$, if and only if they are composed of the same simplices. Two n-simplicial complexes $\Sigma_i$ and $\Sigma_j$, which are s-adjacent along an s-simplex $\sigma$, are said to be strongly matching along $\sigma$ if and only if

- $\sigma \cap V_i \equiv \sigma \cap V_j$;

- Let $V_{ij} = \sigma \cap V_i \equiv \sigma \cap V_j$ and let $\Sigma_r, r = i,j$, be the s-complex whose s-skeleton is $V_{ij}$ such that $D(\Sigma_r) \equiv \sigma$ and composed of simplices of $\Sigma_r$. Then $\Sigma_r \equiv \Sigma_r$.

Figure 3 shows examples of weakly and strongly matching simplicial complexes. It has been proven [Ber93] that, if $\Sigma_i$ and $\Sigma_j$ are two strongly matching simplicial complexes, then they are weakly matching as well.
Let $S = \{\Sigma_0, \ldots, \Sigma_k\}$ be a family of $n$-simplicial complexes such that $\forall i = 1, \ldots, h, \exists \sigma_i \in \bigcup_{j=0}^{h} \Sigma_j^{(n)}$, with $\sigma_i \equiv D(\Sigma_i)$. A hierarchy of simplicial complexes of order $n$ is a family $S$ described by a tree $\mathcal{H} = (S, \mathcal{E})$, where $\mathcal{E} = \{(\Sigma_i, \Sigma_j) \mid \Sigma_i, \Sigma_j \in S \text{ and } \exists \sigma_j \in \Sigma_j^{(n)} \text{ such that } \sigma_j \equiv D(\Sigma_j) \text{ and } \Sigma_j^{(n)} \neq \{\sigma_j\}\}$. Each arc $(\Sigma_i, \Sigma_j) \in \mathcal{E}$ is labeled with $\sigma_j$.

A Generalized Hierarchical Simplicial Complex (GHSC) of order $n$ is a hierarchy of simplicial complexes $\mathcal{H} = (S, \mathcal{E})$ of order $n$ in which any two simplicial complexes $\Sigma_i$ and $\Sigma_j$ of $\mathcal{H}$ adjacent along an $s$-simplex $\sigma$ ($1 \leq s < n$) are weakly matching along $\sigma$.

A Hierarchical Simplicial Complex (HSC) of order $n$ is a hierarchy of simplicial complexes $\mathcal{H} = (S, \mathcal{E})$ of order $n$ satisfying the following two conditions:

(i) for every pair $\Sigma_i, \Sigma_j$ of complexes of $S$, adjacent along an $s$-simplex $\sigma$ ($1 \leq s \leq n - 1$), with $V_i$ and $V_j$ 0-skeletons of $\Sigma_i$ and $\Sigma_j$, respectively, one of the following conditions must hold:

- $\Sigma_i$ and $\Sigma_j$ are strongly matching along $\sigma$;
- $\Sigma_i$ and $\Sigma_j$ are weakly matching along $\sigma$, with $V_i \cap \sigma \subseteq V_j \cap \sigma$ and there exists a descendant of $\Sigma_i$ which is strongly matching with $\Sigma_j$ along $\sigma$;
- $\Sigma_i$ and $\Sigma_j$ are weakly matching along $\sigma$, with $V_j \cap \sigma \subseteq V_i \cap \sigma$ and there exists a descendant of $\Sigma_j$ which is strongly matching with $\Sigma_i$ along $\sigma$;

(ii) for every unexpanded $n$-simplex $\sigma_i$ and for every macrosimplex $\sigma_j$ adjacent to $\sigma_i$ along an $s$-simplex $\sigma_p$, for all descendant simplices $\sigma_r$ of $\sigma_j$ such that $\sigma_r \cap \sigma_p \neq \emptyset$ and $\sigma_r \cap \sigma_p$ is not on the boundary of $\sigma_p$, then $\sigma_r \cap \sigma_p \equiv \sigma_p$.

Condition (ii) simply means that all macrosimplices, which are $(n - 1)$-adjacent to an unexpanded $n$-simplex $\sigma$, must not modify the face they have in common with $\sigma$. The strongly matching rule, which characterizes an HSC, ensures that each macrosimplex of the subtree rooted at a simplicial complex $\Sigma_k$ is expanded into a complex whose boundary faces are shared by its adjacent complexes.

Note that the class of the HSCs is a proper subclass of that of the GHSCs. It has been proven that the expanded complex associated with a GHSC is a generalized simplicial complex, while the one associated with an HSC is always a simplicial complex [Ber93]. On the other hand, partially expanded complexes associated with an HSC can either be simplicial complexes or generalized ones.

Because of its importance as a basis for a surface model, we have considered a special class of GHSCs and HSCs, i.e., those based on Delaunay simplices. A Delaunay $n$-complex $\Sigma$ of a set of points $V \in \mathbb{R}^n$ is a convex, $(n-1)$-connected simplicial complex such that the circumsphere of any simplex $\sigma \in \Sigma$ does not contain any vertex of $V$ inside. A (generalized) hierarchical simplicial complex, in which each component is a Delaunay simplicial complex, is called a Delaunay HSC (GHSC).

5 A cyclicity of Hierarchical Cell Complexes

In this section, we show that any (partially or fully) expanded complex associated with a hierarchical cell complex, in which each complex is acyclic as well, is acyclic, by extending the result by Edelsbrunner on the acyclicity of $n$-dimensional projective cell complexes [Ede60].

An $n$-dimensional HCC $\mathcal{H}$ is said to be acyclic if and only if each of its component is acyclic. Theorem 5.3 relates the acyclicity of $\mathcal{H}$ with that of the expanded complex associated with it. To prove Theorem 5.3, we need a result which ensures the acyclicity of a complex generated through the refinement process. Lemma 5.1 shows that, if an $n$-cell $\gamma_2$ of an $n$-complex $\Gamma_i$ obstructs one of the $n$-cells belonging to the direct expansion of an $n$-cell $\gamma_1$ of $\Gamma_i$ $(n-1)$-adjacent to $\gamma_2$, then $\gamma_2$ obstructs $\gamma_1$.

Lemma 5.1. Let $v_p$ be a viewpoint. Let $\Gamma_i$ be an acyclic cell complex of order $n$. Let $\gamma_1, \gamma_2 \in \Gamma_i^{(n)}$ such that $\gamma_1$ and $\gamma_2$ are $(n-1)$-adjacent. Let $\Gamma_i$ be an $n$-complex such that $\gamma_1 \equiv D(\gamma_2)$. Let $\Gamma_i$ be the cell complex obtained by refining $\gamma_1$ in $\Gamma_i$ with $\Gamma_i$. If there exists an $n$-cell $\gamma_k \in \Gamma_k^{(n)}$ such that $\gamma_k < v_p \gamma_2$, then $\gamma_1 < v_p \gamma_2$.

Proof. Since $\gamma_k < v_p \gamma_2$, there exists a ray $r_k$ emanating from $\gamma_k$ such that $\gamma_k \cap \mathcal{V}_k \neq \emptyset$ and $\gamma_k \cap \mathcal{V}_k = \emptyset$ and $\forall p_k \in r_k \cap \mathcal{V}_k$, $p_k$ lies between $v_p$ and $p_k$, $\forall p_k \in r_k \cap \mathcal{V}_k$. Let $p_k \in r_k \cap \mathcal{V}_k$ and $p_k \in r_k \cap \mathcal{V}_k$; $p_k$ lies between $v_p$ and $p_k$. Let us assume that $\gamma_k < v_p \gamma_1$. As $\Gamma_i$ is acyclic, $\forall \mathcal{V}_r$, such that $\mathcal{V}_r \cap (v_p) \neq \emptyset$ and $\mathcal{V}_r \cap (v_p) \neq \emptyset$ we have that $\forall p \in \mathcal{V}_r$, $p$ lies between $v_p$ and $p_k$. But there exists ray $r_k$ such that $r_k \cap (v_p) \neq \emptyset$ (since $r_k \cap (v_p) \subseteq r_k \cap (v_1)$) and $r_k \cap (v_p) \neq \emptyset$ and, moreover, $p_k \in r_k \cap (v_1)$, does not lie between $v_p$ and $p_2 \in r_k \cap (v_p)$.

Similarly, we can prove that if an $n$-cell $\gamma_2$ of an $n$-complex $\Gamma_i$ is obstructed by one of the $n$-cells belonging to the direct expansion of an $n$-cell $\gamma_1$ of $\Gamma_i$ $(n-1)$-adjacent to $\gamma_2$, then $\gamma_1$ obstructs $\gamma_2$.

Lemma 5.2 shows that the refinement of an $n$-cell in an acyclic $n$-complex with an acyclic $n$-complex preserves the infront/behind relation.
Lemma 5.2 Let $\Gamma_i$ and $\Gamma_j$ be two cell complexes of order $n$. Let $\gamma_j \in \Gamma_i^{(n)}$. Let $\Gamma_i \rightarrow \Gamma_j$ be the cell complex obtained by refining $\gamma_j$ in $\Gamma_i$ with $\Gamma_j$. If $\Gamma_i$ and $\Gamma_j$ are acyclic, then $\Gamma_i \rightarrow \Gamma_j$ is acyclic.

Proof. Let us assume that there exists a simple cycle $S = \{\gamma_1, \ldots, \gamma_z\}$ in $\Gamma_i \rightarrow \Gamma_j$, i.e., for some viewpoint $v_p$, $\gamma_1 < v_p \gamma_2 < v_p \ldots < v_p \gamma_z < v_p \gamma_1$. Cells $\gamma_1, \ldots, \gamma_z$ do not all belong to $\Gamma_i$ or to $\Gamma_j$ (otherwise either $\Gamma_i$ or $\Gamma_j$ would not be acyclic). Let us now consider the ordered sequence $S$ obtained from $S$ by substituting all the cells of $\Gamma_j$ with $\gamma_j$. $S$ defines a cycle of cells, all belonging to $\Gamma_i$, because of Lemma 5.1. In fact, let $\gamma_k, \gamma_{k+1} \in S$ such that $\gamma_k \in \Gamma_i$ and $\gamma_{k+1} \in \Gamma_j$; if $\gamma_k < v_p \gamma_{k+1}$, then $\gamma_k < v_p \gamma_j$. This contradicts the hypothesis that $\Gamma_i$ is acyclic. Hence, $\Gamma_i \rightarrow \Gamma_j$ must be acyclic. □

Theorem 5.3 Let $\mathcal{H} = (N, E)$ be a hierarchical cell complex of order $n$ such that $N = \{\Gamma_0, \Gamma_1, \ldots, \Gamma_n\}$, rooted at $\Gamma_0$. If $\mathcal{H}$ is acyclic, then the fully expanded cell complex $\Gamma_\mathcal{H}$ associated with $\mathcal{H}$ is acyclic.

Proof. We show that if, for every $i = 0, \ldots, h$, $\Gamma_i$ is acyclic, then $\Gamma_\mathcal{H}$ is acyclic. Let $\mathcal{H}'$ be a hierarchical subcomplex of $\mathcal{H}$ rooted at $\Gamma_0$. The proof is done by induction on the number of nodes of $\mathcal{H}'$. Let $\Gamma_\mathcal{H}'$ be the expanded complex associated with $\mathcal{H}'$. We must prove that $\Gamma_\mathcal{H}'$ is acyclic. If $\mathcal{H}'$ has only one node, then $\Gamma_\mathcal{H}'$ is the same as $\Gamma_0$, which is acyclic. We assume that every hierarchical subcomplex of $\mathcal{H}$ rooted at $\Gamma_0$ with less than $s$ nodes has an expanded complex associated which is acyclic. Let $\mathcal{H}''$ be one of such subcomplexes and $\Gamma_{\mathcal{H}''}$ the expanded complex associated with it, and let $\mathcal{H}'$ be the hierarchical subcomplex of $\mathcal{H}$ with exactly $s$ nodes such that $\Gamma_{\mathcal{H}''}$ corresponds to the refinement of a macrocell in $\Gamma_{\mathcal{H}''}$. To prove that $\Gamma_\mathcal{H}'$ is an acyclic complex, it is enough to show that the process of refining a macrocell $\gamma_j$ in an acyclic complex with another acyclic complex preserves acyclicity. This is a consequence of Lemma 5.2. □

The following result is an immediate consequence of Theorem 5.3.

Corollary 5.4 Let $\mathcal{H} = (N, E)$ be an acyclic hierarchical cell complex of order $n$. Any partially expanded complex associated with $\mathcal{H}$ is acyclic.

A visibility order of the cells in a cell complex resulting from the expansion of a hierarchical cell complex can be obtained by combining a tree traversal with a topological sorting algorithm applied to each complex, as shown in [Ber93].

6 Concluding Remarks

We have presented a new hierarchical representation for cell complexes in $n$ dimensions, called a hierarchical cell complex. We have also defined hierarchical simplicial complexes for a multiresolution representation of a hypersurface in $E^{n+1}$. In particular, we are interested in the application of such representation to volume data description at different levels of resolution. An important requirement in such application is the capability of sorting the cells forming the complex, used as basic domain discretization, in a back-to-front order. To this aim, we have proven the acyclicity of the inflow/behind relation on any partially or fully expanded complex associated with an HCC composed of acyclic cell complexes. Thus, an HCC can be ordered back-to-front at any level of resolution.

Further developments of the work presented in this paper concern the implementation of a hierarchical simplicial complex in three dimensions for describing 3D surfaces at increasingly higher level of resolution, and the development of algorithms for isosurface extraction and direct volume rendering from the resulting model.

References


