Planar Visibility Graphs

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ABSTRACT

The recognition problem for visibility graphs is, given a graph, to determine whether this graph is the visibility graph of a simple polygon. It is not known whether this problem could be solved in polynomial time. It is even not known whether the recognition problem for planar visibility graphs could be solved in polynomial time. In this paper, we shall give the necessary and sufficient conditions for a polygon to have a planar visibility graph. Using this result, we can prove that no 4-connected planar graph could be a visibility graph of a simple polygon.

INTRODUCTION AND TERMINOLOGY

All polygons discussed in this paper are assumed simple. Two vertices of a polygon $P$ are said visible from each other (or they can see each other) if the closed line segment joining them does not intersect the exterior of $P$. (Note that the line segment may touch the boundary of $P$ several times.) The visibility graph of a polygon $P$, denoted by $G_p$, is the graph whose vertices correspond to the vertices of $P$ and two vertices of $G_p$ are joined by an edge if and only if their corresponding vertices in $P$ are visible.

O'Rourke [2] believes that "some of the fundamental unsolved problems involving visibility in computational geometry will not be solved until the combinatorial structure of visibility is more fully understood. Perhaps the purest condensation of this structure is a visibility graph."
A necessary condition for a graph to be a visibility graph is the existence of at least one Hamiltonian cycle, corresponding to the boundary of the polygon. In [4], Tutte proved that every 4–connected planar graph has a Hamiltonian cycle. In this paper, we shall give the necessary and sufficient conditions for a polygon to have a planar visibility graph. Using this result, we can also prove that no 4–connected planar graph could be a visibility graph; this property has also been derived by Abello, Lin and Pisupati [1]. For literature of the visibility problem, please refer to [3].

The following terminologies will be used throughout this paper. Let \( P \) be a polygon. Two vertices \( U \) and \( V \) of \( P \) will divide the boundary of \( P \) into two chains: one is called \( UV\text{–chain} \) and the other, \( VU\text{–chain} \). \( UV\text{–chain} \) contains those vertices between \( U \) and \( V \) (with \( U \) and \( V \) being excluded) when the boundary of \( P \) is traversed clockwise from \( U \) to \( V \). \( VU\text{–chain} \) is defined similarly. We shall use \( \Delta ABC \) to denote a triangle formed by three vertices \( A, B, C \) of \( P \). A triangle \( \Delta ABC \) is called an empty triangle if \( A, B, C \) can see each other and no other vertex of \( P \) lies entirely within \( \Delta ABC \) (it may lie on the boundary of \( \Delta ABC \)).

The three vertices of an empty triangle \( \Delta ABC \) will divide the boundary of a polygon \( P \) into three chains: \( AB\text{–chain} \), \( BC\text{–chain} \), and \( CA\text{–chain} \). For convenience, in the following discussion when we say "the three chains of \( \Delta ABC \)," we mean \( AB\text{–chain} \), \( BC\text{–chain} \), and \( CA\text{–chain} \). When we say "\( AB\text{–chain} \) can see \( BC\text{–chain} \)," we mean there is a vertex on \( AB\text{–chain} \) which can see a vertex on \( BC\text{–chain} \). "\( BC\text{–chain} \) can see \( CA\text{–chain} \)" and "\( CA\text{–chain} \) can see \( AB\text{–chain} \)" are defined similarly.

**PROPERTIES OF A POLYGON WHICH HAS A PLANAR VISIBILITY GRAPH**

Let \( P \) be a polygon and let \( \Delta ABC \) be an empty triangle of \( P \). If \( G_p \) is planar, we can prove that \( P \) satisfies the following properties.

**Property 1.** Each of the three chains of \( \Delta ABC \) has at most one vertex that can see all of \( A, B, C \). (Otherwise, \( G_p \) contains a subdivision of \( K_5 \).)

**Property 2.** \( P \) has at most two vertices that can see all of \( A, B, C \). (Otherwise, \( G_p \) contains a subdivision of \( K_{3,3} \)).
Property 3. If two of the three chains of $\triangle ABC$ can see each other, then these two chains have exactly one vertex that can see all of $A$, $B$, $C$.

Property 4. If one of the three chains of $\triangle ABC$ can see both of the other two chains of $\triangle ABC$, then $P$ has exactly one vertex $V$ that can see all of $A$, $B$, $C$, and $V$ lies on the chain which can see both of the other two chains.

Property 5. It is impossible for all of the three chains of $\triangle ABC$ to see each other.

**MAIN RESULTS**

Let $P$ be a simple polygon which has a planar visibility graph and let $\triangle ABC$ be an empty triangle of $P$. We shall introduce six possible structures of $\triangle ABC$, denoted as structure (0), structure (1a), structure (1b), structure (1c), structure (2a) and structure (2b). See Figure 1 as an illustration.

Structure (0). There is no vertices on the three chains of $\triangle ABC$ that can see all of $A$, $B$, $C$.

(Note that in this case, the three chains can not see each other. Otherwise, from Property 3, there is a vertex can see all of $A$, $B$, $C$.)

Structure (1a). There is only one vertex $V$ on the three chains of $\triangle ABC$ that can see all of $A$, $B$, $C$, and all the three chains can not see each other. (For example, in Figure 1, $P$ has only one vertex $V$ that can see all of $A$, $B$, $C$, and $V$ is on $AB$–chain.)

Structure (1b). There is only one vertex $V$ on the three chains of $\triangle ABC$ that can see all of $A$, $B$, $C$, and only two of the three chains can see each other. (For example, in Figure 1, $AB$–chain and $BC$–chain can see each other. From Property 3, $V$ is on $AB$–chain or $BC$–chain. In this example, $V$ is on $AB$–chain.)

Structure (1c). There is only one vertex $V$ on the three chains of $\triangle ABC$ that can see all of $A$, $B$, $C$, and only one of the three chains can see the other two chains. (For example, in Figure 1, $AB$–chain can see both $BC$–chain and $CA$–chain. From Property 4, $V$ is on $AB$–chain.)

Structure (2a). There are two vertices $U$ and $V$ on the three chains of $\triangle ABC$ that can see all of $A$, $B$, $C$, and all the three chains can not see each other. (From Property 1, $U$
and \( V \) lie on different chains. In Figure 1, \( U \) is on \( AB \)-chain and \( V \) is on \( BC \)-chain.)

**Structure (2b).** There are two vertices \( U \) and \( V \) on the three chains of \( \Delta ABC \) that can see all of \( A, B, C \), and only two of the three chains can see each other. (Note that \( U \) can not see \( V \); otherwise, \( G_p \) contains a subdivision of \( K_5 \).) (For example, in Figure 1, \( AB \)-chain and \( BC \)-chain can see each other. From Property 3, only one of \( U \) and \( V \) is on \( AB \)-chain and \( BC \)-chain. In this example, \( U \) is on \( AB \)-chain and hence \( V \) is on \( CA \)-chain.)

**Theorem 1.** \( G_p \) is planar if and only if every empty triangle \( \Delta ABC \) of \( P \) has structure (0), structure (1a), structure (1b), structure (1c), structure (2a) and structure (2b).

**Proof:** *Necessity:* From Property 2, there are only three possible cases:

(i) There is no vertex on the three chains of \( \Delta ABC \) that can see all of \( A, B, C \). In this case, \( \Delta ABC \) has structure (0).

(ii) There is only one vertex on the three chains of \( \Delta ABC \) that can see all of \( A, B, C \). In this case, if \( G_p \) is planar, then from Property 3, Property 4, and Property 5, \( \Delta ABC \) has structure (1a), structure (1b), or structure (1c).

(iii) There are two vertices on the three chains of \( \Delta ABC \) that can see all of \( A, B, C \). In this case, if \( G_p \) is planar, then from Property 1, Property 3, Property 4, and Property 5, \( \Delta ABC \) has structure (2a) or structure (2b).

Therefore if \( G_p \) is planar, every empty triangle of \( P \) has one of the above six structures.

*Sufficiency:* We can prove by induction on the number of vertices of \( P \) that if every empty triangle of \( P \) has one of the above six structures, then we can find a planar embedding for \( G_p \). Hence \( G_p \) is planar. This part of proof is omitted here. Q.E.D.

Every planar visibility graph must be 2-connected. It is not difficult to construct 3-connected planar visibility graphs. A necessary condition for a graph to be a visibility graph of a polygon is the existence of at least one Hamiltonian cycle, corresponding to the boundary of the polygon. In [4], Tutte proved that every 4-connected planar graph must have a Hamiltonian cycle. However, we can use Theorem 1 to prove the following theorem.
Theorem 2. No 4-connected planar graph could be a visibility graph.

The proof of Theorem 2 is omitted here. This property was also derived by Abello, et al. [1] using a different approach. We believe that the six structures proposed in this paper will be useful for completely solving the recognition problem for planar visibility graphs.

REFERENCES

Figure 1