Finding Hamiltonian Circuits in Arrangements of Jordan Curves is NP-Complete

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Abstract

Let $A = \{C_1, C_2, \ldots, C_n\}$ be an arrangement of Jordan curves in the plane lying in general position, i.e., every curve properly intersects at least one other curve, no two curves touch each other and no three meet at a common intersection point. The Jordan-curve arrangement graph of $A$ has as its vertices the intersection points of the curves in $A$, and two vertices are connected by an edge if their corresponding intersection points are adjacent on some curve in $A$. We further assume $A$ is such that the resulting graph has no multiple edges. Under these conditions it is shown that determining whether Jordan-curve arrangement graphs are Hamiltonian is NP-complete.

1 Introduction

A Hamiltonian circuit in a graph is a circuit which passes through every vertex of the graph exactly once. The Hamiltonian circuit problem asks whether there exists at least one Hamiltonian circuit in a given graph. There have been at least three approaches taken in the past towards the study of Hamiltonian circuits. In one approach sufficient conditions are sought for which graphs are Hamiltonian. For example, it is known that all 4-connected triangulated graphs [Wh31], 4-connected planar graphs [Tu56], [Ch85] and 1-sail line arrangement graphs [EGT92] are Hamiltonian. Also, the visibility graphs of sets of line segments with the property that the line segments are of unit length whose endpoints have integer coordinates, are Hamiltonian [OR91]. A related computational question concerns how fast we can find a Hamiltonian circuit in a Hamiltonian graph. For any 4-connected planar graph $G$ with $n$ vertices, a Hamiltonian circuit in $G$ can be found in $O(n^3)$ time [Go82]. If only the vertices where a turn is made need be reported (a streamlined Hamiltonian circuit) then a Hamiltonian circuit for 1-sail line arrangement graphs can be found in $\Theta(n \log n)$ time, where $n$ is the number of lines in the arrangement [EGT93]. A second approach is to find restricted classes of graphs for which we can determine in polynomial time whether or not instances of such graphs admit a Hamiltonian circuit. For example if each line segment of a set of $n$ disjoint line segments in the plane has at least one of its end points on the convex hull of the set, it
2 Definitions and Results

Let $C_i$ and $C_j$ denote two Jordan curves. Let $A = \{C_1, C_2, \ldots, C_n\}$ be an arrangement of Jordan curves. The set of intersection points of $C_i$ and $C_j$ is denoted by $I(C_i, C_j)$, and the set of intersection points has measure zero. In this paper, we assume that (i) no two curves touch each other (thus, $|I(C_i, C_j)| > 1$ if $C_i$ properly intersects $C_j$) and (ii) no three curves are concurrent, i.e., share a common intersection point. Let $V(A) = \{v \mid v \in I(C_i, C_j) \text{ such that } C_i, C_j \in A\}$. The Jordan-curve arrangement graph of an arrangement $A$ is the graph $G_a = (V_a, E_a)$ such that (i) $V_a = V(A)$ and (ii) edges in $E_a$ are formed by curves of $A$ (see Fig. 1). Note that a Jordan-curve arrangement graph may contain multi-edges.

**Theorem 1.** The Hamiltonian circuit problem for Jordan-curve arrangement graphs with no multi-edges is NP-complete.

**Remark.** Since every Jordan-curve arrangement graph is a 4-regular planar graph, the Hamiltonian circuit problem for 4-regular planar graphs is also NP-complete. The class of Jordan-curve arrangement graphs with no multi-edges is properly contained in the class of 4-regular planar graphs, i.e., there exist 4-regular planar graphs $G$ with no multi-edges such that $G$ cannot be formed by any arrangement of Jordan curves (see Fig. 2).

**Proof of Theorem 1.** Since the Hamiltonian circuit problem for general graphs is in NP [Ka72], the problem for Jordan-curve arrangement graphs with no multi-edges is also in NP. It is known that the Hamiltonian circuit problem for 3-regular planar graphs with no multi-edges is NP-complete [GJT76]. We reduce each 3-regular planar graph $G$ with no multi-edges to a Jordan-curve arrangement graph $G_a$ with no multi-edges such that $G$ is Hamiltonian if and only if $G_a$ is Hamiltonian. The overview of the proof is as follows. Starting with $G$, (i) we construct 4-regular planar graph
\(G_1\) with \textit{multi}-edges and (ii) we then construct 4-regular planar graph \(G_a\) with no multi-edges. (iii) We prove that \(G\) is Hamiltonian if and only if \(G_a\) is Hamiltonian (Lemma 1), and (iv) we then prove that \(G_a\) is a Jordan-curve arrangement graph (Lemma 2).

Construction of \(G_1\): We first replace each edge of \(G\) by a pair of multi-edges (see Fig. 3). We then replace each vertex \(v\) by three vertices (say \(v_x, v_y,\) and \(v_z\)) and three edges, \((v_x, v_y), (v_y, v_z), \) and \((v_z, v_x)\). We call the subgraph induced by these three edges a \textit{triangle}. We denote the resulting graph by \(G_1 = (V_1, E_1)\). By construction, \(G_1\) is a 4-regular planar graph with \textit{multi}-edges. Furthermore, \(G_1\) can be constructed from \(G\) in polynomial time.

Construction of \(G_a\): The basic idea is to add four vertices and four edges to each pair of multi-edges (see Figs. 4 and 5). We first find a vertex subset \(S \subseteq V_1\) such that (i) exactly one vertex of each pair of multi-edges is in \(S\) and (ii) at least one vertex of each triangle is in \(S\). (We will show how to find such an \(S\) later.) Suppose that a vertex \(a\) of \(G_1\) is in \(S\) (see Fig. 4-(a)). Let \(e_1, e_2, e_3,\) and \(e_4\) be the edges incident to \(a\) in clockwise order. For \(1 \leq i \leq 4\), we “divide” edge \(e_i\) into two edges by adding a new vertex \(a_i\) on \(e_i\) (see Fig. 4-(b)). We then add four edges \((a_1, a_2), (a_2, a_3), (a_3, a_4),\) and \((a_4, a_1)\). We call a subgraph induced by these four edges a \textit{circle}. By applying the above procedure to each vertex in \(S\), we obtain \(G_a\) (see Fig. 5-(b)). Now \(G_a\) has no multi-edges.

It remains to show how to construct \(S \subseteq V_1\) in polynomial time. We first construct a vertex set \(S_1 \subseteq S\) such that (i) at most one vertex of each pair of multi-edges is in \(S_1\) and (ii) \textit{exactly} one vertex of each triangle is in \(S_1\). \(S\) can be obtained by finding pairs of multi-edges such that both vertices of each of the pairs are not in \(S_1\) and by adding one arbitrary vertex of each such pair to \(S_1\). The construction of \(S_1\) is as follows. We construct a \textit{directed} subgraph \(D = (V, E_D)\) in the original graph \(G = (V, E)\) such that all of \(D\)’s vertices have out-degree one (see Fig. 5-(a)). Recall that vertices and edges in \(G\) were replaced by triangles and pairs of multi-edges in \(G_1\), respectively (see Fig. 3-(b)). Each vertex \(v_x\) of \(G_1\) is in \(S_1\) if and only if there exists a directed edge \((v, x)\) in \(E_D\) (see Figs. 3-(b) and 5-(a)). \(D = (V, E_D)\) can be constructed as follows. We first find an undirected spanning tree, say \(T = (V, E_T)\), in \(G\). We choose an arbitrary vertex, say \(r\), among \(T\)’s leaves. We regard \(r\) as the new root of \(T\). We then construct a directed spanning tree rooted at \(r\) by adding direction information to \(T\). Note that every vertex of \(T\), except for root \(r\), now has outdegree exactly one. \((r\) has in-degree one and out-degree zero.) Furthermore, we find an undirected edge \((r, x) \in E\) such that \((x, r) \notin E_T\). By adding \textit{directed} edge \((r, x)\) to \(E_T\), we obtain \(E_D\) (and hence we obtain \(D = (V, E_D)\)).

\textbf{Lemma 1.} \(G\) is Hamiltonian if and only if \(G_a\) is Hamiltonian.

\textbf{Proof.} (\(\Leftarrow\)) Let \(v\) be a vertex in the 3-regular planar graph \(G\) (See Fig. 3-(a)). By the above reduction, vertex \(v\) is reduced into a subgraph, say \(S_v\), composed of one triangle and at least one circle (see Fig. 6). It should be noted that removing three vertices, \(v_1, v_2,\) and \(v_3\) (shown in Fig. 6), from \(G_a\) decomposes \(G_a\) into at least two connected components, one of which corresponds to \(v\) in \(G\). (Intuitively, \(v_1, v_2,\) and \(v_3\) in Fig. 6 correspond to the three edges \((v, x), (v, y),\) and \((v, z)\) in Fig. 3-(a), respectively.) If there is a Hamiltonian circuit in \(G_a\) which passes through vertices in \(S_v\) from \(v_1\) to \(v_2\), then we can construct the corresponding Hamiltonian circuit in \(G\) which passes through \(v\) from \((x, v)\) to \((v, y)\). Therefore, if there is a Hamiltonian circuit in \(G_a\), then there is a Hamiltonian circuit in \(G\).

(\(\Rightarrow\)) For each of the possible cases shown in Fig. 6, \(S_v\) has a Hamiltonian path between every two of the three vertices, \(v_1, v_2,\) and \(v_3\) (see Fig. 7, symmetric cases are omitted). Thus, if there is a Hamiltonian circuit in \(G\), then we
can construct the corresponding Hamiltonian circuit in $G_a$. □

**Lemma 2.** $G_a$ is a Jordan-curve arrangement graph.

**Proof.** Since $G_a$ was constructed by adding circles to $G_1$, $G_a$ is a Jordan-curve arrangement graph if $G_1$ is a Jordan-curve arrangement graph. In the following, we show $G_1$ is a Jordan-curve arrangement graph. Consider the following edge-coloring algorithm:

1. Initially, all edges have no colors.
2. Choose an arbitrary edge with no color, and color it with a new color.
3. Find an edge, say $e_1$, with no color which satisfies the following condition: There exist three edges $e_2, e_3$ and $e_4$ such that $e_1, e_2, e_3$, and $e_4$ are incident to a vertex in clockwise order and that $e_3$ has already been colored. (See Fig. 4-(b).)
4. Color $e_1$ with the same color as $e_3$.
5. Repeat (3) and (4) until there is no edge $e_1$ satisfying the above condition.
6. Repeat (2)-(5) until all edges are colored.

By applying this algorithm to $G_1$, we obtain subgraphs each of which consists of edges having the same color. We now show that these subgraphs are 2-regular graphs.

Consider an arbitrary face of $G$ (see Fig. 8-(a)). By the transformation from $G$ to $G_1$, (i) each vertex of $G$, which is the boundary point of three faces, is replaced by three edges of a triangle of $G_1$, and (ii) each edge of $G$, which is the boundary between two faces, is replaced by a pair of multi-edges (see Fig. 8-(b)). Thus, the edges colored by a single color form a 2-regular planar subgraph which corresponds to a face of $G$. Hence, $G_1$ can be formed by an arrangement of Jordan curves. □

**Example.** We give an example of arrangements of Jordan curves whose graphs are not Hamiltonian (see Fig. 9-(c)). The 3-regular planar graph shown in Fig. 9-(a) is not Hamiltonian, since it is not 1-tough [Ch85], i.e., we can decompose it into three connected components by removing two vertices. This non-Hamiltonian graph can be reduced to the 4-regular planar graph with no multi-edges shown in Fig. 9-(b). This 4-regular graph is also non-Hamiltonian, since removing two subgraphs which correspond to the above two vertices decomposes the 4-regular graph into three connected components. (Although there is a Hamiltonian path in each of the three connected components, no Hamiltonian circuit can be constructed by connecting those three Hamiltonian paths.) Therefore, the arrangement of Jordan curves shown in Fig. 9-(c) is non-Hamiltonian.

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**References**


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Fig. 6 Three possible cases

Fig. 7 Hamiltonian paths

Fig. 8 (a) Face of $G$ (b) Corresponding 2-regular subgraph in $G_1$

Fig. 9 (a) 3-regular graph (b) 4-regular graph with no multi-edges (c) Arrangement of Jordan curves

Fig. 9 Non-Hamiltonian planar graphs and a non-Hamiltonian arrangement