Edge guarding a triangulated polyhedral terrain

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Abstract: In this note we show that \( \lceil n/3 \rceil \) guards are always sufficient to guard a triangulated polyhedral terrain on \( n \) vertices. This is equivalent to showing that \( \lceil n/3 \rceil \) edges are sufficient to cover all of the faces of a planar triangulation on \( n \) vertices.

1 Introduction

The art gallery problem, originally posed in 1973, is to determine the minimum number of guards sufficient to cover the interior of a simple \( n \)-sided polygon. In 1975 Chvatal resolved the problem showing that \( \lceil n/3 \rceil \) guards are always sufficient. Since this time many variants of the art gallery problem have been studied [O87]. Here we consider the variant in which the guards are permitted to patrol along the edges of the polyhedral terrain they wish to guard.

A polyhedral terrain is a polyhedral surface in three dimensions such that the intersection of the terrain with a vertical line is either empty or a point. A polyhedral terrain is triangulated if each of its faces is a triangle. Two points \( x \) and \( y \) of a terrain are said to be visible if the line segment \( xy \) does not contain any points below the terrain. A point \( x \) of a terrain is said to be visible to an edge \( e \) if there exists a point \( y \) on \( e \) such that \( x \) and \( y \) are visible. A set of edges is said to guard a terrain if every point of the terrain is visible from one of the edges. We call the problem of finding such a set of edges the terrain edge guarding problem. It has been shown in [BSTZ92] that \( \lceil (4n - 4)/13 \rceil \) edges are sometimes necessary to guard a terrain. It is the purpose of this note to establish that \( \lceil n/3 \rceil \) edges are always sufficient.

Let \( G=(V,E) \) be a planar triangulated graph on \( n \) vertices. A set of edges \( H \) in \( G \) is said to guard \( G \) if every face of \( G \) contains at least one vertex in the vertex set of \( H \). We call the problem of finding such a set of edges the combinatorial edge guarding problem. It is easy to see that a solution to the combinatorial edge guarding problem is also a solution to the terrain edge guarding problem: associate to a given a terrain \( T \) a planar triangulated graph \( G(T) \) corresponding to the projections of the vertices and edges of \( T \) onto a horizontal plane lying below \( T \). In this note we show that \( \lceil n/3 \rceil \) edges are always sufficient to guard a planar triangulated graph on \( n \) vertices and the result for terrains follows.

We need the following definitions. A coloring of a graph is an assignment of colors to vertices such that no two adjacent vertices receive the same color. The Four-color Theorem states that any planar graph can be colored by at most four colors [AH77]. A matching is a subset \( M \) of the edges of a graph such that no vertex is contained in more than one edge of \( M \). A matching \( M \) is called maximal if no other edge can be added to \( M \) such that it remains a matching. The size of a matching is the number of edges in it. We note here that given a planar graph, a four-coloring can be found in time \( O(n^2) \) [AH77] and a maximal matching in linear time using a greedy algorithm.

2 Main Theorem

**Theorem:** Every planar triangulation \( G \) on \( n \) vertices can be guarded with \( \lceil n/3 \rceil \) edges.

**Proof:** Let \( \{c_1, c_2, c_3, c_4\} \) be the set of four colors used in a coloring of \( G \) and let \( v_1, v_2, v_3 \), and \( v_4 \) be the sets of vertices colored by \( c_1, c_2, c_3 \), and \( c_4 \) respectively. Notice that since \( G \) is triangulated, each face contains three vertices colored by three distinct colors and consequently, any pair of color classes \( \{v_i, v_j\}, 1 \leq i < j \leq 4 \), contains a vertex from each face. Let \( G_{ij} \) be the subgraph of \( G \) induced by \( v_i \) and \( v_j \) and let \( M_{ij} \) be a maximal matching in \( G_{ij}, 1 \leq i < j \leq 4 \). An edge guarding of \( G \) can be construed by taking the edges of \( M_{ij} \) plus one edge incident to each vertex in \( v_i \cup v_j \) that is not in any edge of \( M_{ij} \). The size of this edge guarding is given by \( |v_i| + |v_j| - |M_{ij}| \). The average size of \( |v_i| + |v_j| - |M_{ij}| \) over all \( i \) and \( j \) is

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3n - \sum_{1 \leq i < j \leq 4} \frac{M_{ij}}{6} \text{. Thus, if } \sum_{1 \leq i < j \leq 4} M_{ij} \geq n, \text{ then at least one of these edge guardings has size less than } \lceil n/3 \rceil \text{ and we are done; so suppose this is not the case.}
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Consider the sets \( M_{12} \cup M_{34}, M_{14} \cup M_{23} \), and \( M_{13} \cup M_{24} \). We claim that these sets also constitute edge-guardings. We show this for the set \( M_{12} \cup M_{34} \), the argument for the other sets is similar. Suppose there is a face \( f \) that contains no vertex in the vertex set of \( M_{12} \cup M_{34} \). Since each face is colored by three distinct colors, \( f \) must contain either an edge whose vertices are colored by \( c_1 \) and \( c_2 \) or an edge whose vertices are colored by \( c_3 \) and \( c_4 \); assume the former, the argument for the other case is similar. If this edge is not included in \( M_{12} \) then, since the matching is maximal, at least one of the vertices of this edge must be in some edge of \( M_{12} \). But this is a contradiction since we suppose that \( f \) contains no vertex in the vertex set of \( M_{12} \cup M_{34} \). The average size of the sets \( M_{12} \cup M_{34}, M_{14} \cup M_{23} \), and \( M_{13} \cup M_{24} \) is

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\sum_{1 \leq i < j \leq 4} \frac{M_{ij}}{3} \text{. Since from the above we have that } \sum_{1 \leq i < j \leq 4} M_{ij} < n, \text{ at least one of } M_{12} \cup M_{34}, M_{14} \cup M_{23}, \text{ and } M_{13} \cup M_{24} \text{ has size less than } \lceil n/3 \rceil \text{ which completes the proof.} \]
3 Open Problems

A polynomial time algorithm for finding $\lceil n/3 \rceil$ edge guards for a triangulated planar graph (or a polyhedral terrain) follows easily from the proof. Since this algorithm involves four coloring the graph it is not very practical. It would be interesting to find a fast algorithm to solve this problem. Also, there remains a small gap between the upper and lower bounds.

References

