Generalized Guarding and Partitioning for Rectilinear Polygons (Extended Abstract)

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Abstract

A T_k-guard G in a rectilinear polygon P is a tree of diameter k completely contained in P. The guard G is said to cover a point x if x is visible (or rectangularly visible) from some point contained in G. We investigate the function $r(n,h,k)$, which is the largest number of T_k-guards necessary to cover any rectilinear polygon with h holes and n vertices. The aim of this paper is to prove new lower and upper bounds on parts of this function.

In particular, we show the following bounds:

1. $r(n,0,k) \leq \left\lfloor \frac{n}{k+4} \right\rfloor$, *with equality for even k*
2. $r(n,h,1) = \lfloor \frac{3n+4h+4}{16} \rfloor$
3. $r(n,h,2) \leq \lfloor \frac{n}{6} \rfloor$.

These bounds, along with other lower bounds that we establish, suggest that the presence of holes reduces the number of guards required, if $k > 1$. In the course of proving the upper bounds, new results on partitioning are obtained which also have efficient algorithmic versions.

1 Introduction

Given two points x and y in a rectilinear polygon P, the points x and y are called *rectangularly visible*, denoted $x \square y$, if the smallest aligned rectangle $R(x,y)$ spanned by x and y is contained in P. In this paper we study the following (rectangular) visibility problem: Let P be a rectilinear polygon with h holes on n vertices. How can one cover P by T_k-guards? Here, a T_k-guard in P is a tree G that has graph-theoretic diameter k and is rectilinearly embedded in P. The region $V(G)$ covered by such a guard is the set of all points rectangularly visible to G: $V(G) = \{x \in P \mid \exists y \in G \text{ such that } x \square y\}$. A collection $\{G_i\}, i \in I$ of T_k-guards covers P if $\bigcup_{i \in I} V(G_i) = P$.

Let us define the following functions:

$$r(P,k) = min\{p \mid \exists \text{ a set of } p\ T_k\text{-guards that cover } P\}$$

$$r(n,h,k) = max\{r(P,k) \mid P \text{ is a rectilinear polygon with } n \text{ vertices and } h \text{ holes}\}$$

Further, let $g(n,h,k)$ be the function analogous to $r(n,h,k)$ defined for general polygons with the usual visibility notion. The first result concerning these functions is Chvátal's classical Art Gallery Theorem, which in our notation reads $g(n,0,0) = \lfloor \frac{n}{3} \rfloor$. After this result, many combinatorial and algorithmic variations of this problem have been studied; most of these variations can be found in [11] and [12]. For general polygons, it is known that $g(n,0,k) = \left\lfloor \frac{n}{k+3} \right\rfloor$ [13] and $g(n,h,0) = \lfloor \frac{n+h}{3} \rfloor$ [8], [2]. Throughout this paper we use the following non-standard convention: $\lfloor \frac{n}{m} \rfloor$ is the set to be 1 for $0 < n < m$.

In rectilinear polygons the situation is quite different. For instance, for point guards (T_0-guards), it is known that $r(n,h,0) = \lfloor \frac{n}{4} \rfloor$ [9], [7]. This is unusual in that the number of holes does not affect the maximum number of guards required. However, for line guards (T_1-guards) holes make the problem harder: it is known that $r(n,h,1) \geq \lfloor \frac{3n+4h+4}{16} \rfloor$ [15]. This bound is tight for $h = 0$ (i.e., $r(n,0,1) = \lfloor \frac{3n+4}{16} \rfloor$) [1]. So what is the correct bound for line guards, and what about general T_k-guards? This paper answers the first question and begins to address the second. We remark that all our results on T_{2k}-guards can be interpreted as results on point guards with $(k+1)$-link visibility [12], and vice versa.

We begin with some definitions and coventions. We use the term *(n,h)-polygon* to denote a rectilinear polygon with h holes and a total of n vertices. It is well known

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that one can restrict the attention to polygons in *general position*, i.e. no two reflex vertices can be joined by a horizontal or vertical line segment lying in the interior of the polygon.

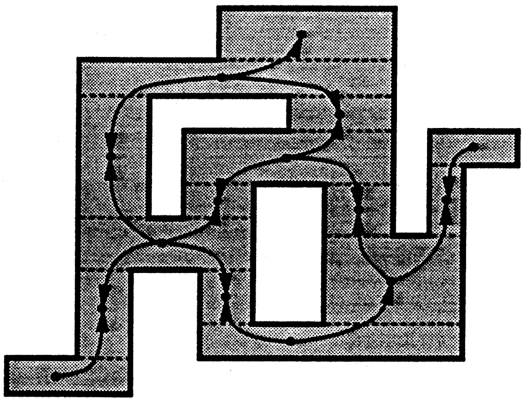

Figure 1: Rectangular decomposition and R-graph

The *rectangular decomposition* of an (n, h)-polygon P is a partition of P into rectangles by extending a horizontal chord into the polygon from every reflex vertex (see Figure 1). The number of rectangles in this decomposition is $\frac{n-2}{2}+h$ (if the polygon were not in general position this number would be smaller). We define the R–graph of P, denoted $\mathbf{R}(P)$ (or simply $\mathbf{R}$ when P is understood), as a directed graph where each vertex corresponds to a rectangle of the rectangular decomposition of P, and an arc is directed from node A to node B iff they correspond to adjacent rectangles and the chord separating these rectangles forms an entire side of B. The direction of these arcs gives us some visibility information. R–graphs are similar to the H–graphs of O'Rourke [10].

The rest of the paper is organized as follows. The next section provides constructions which establish a lower bound for every value of $r(n, h, k)$. The third section contains a proof that $r(n, 0, k) \leq \left\lfloor \frac{n}{k+4} \right\rfloor$, and that equality holds for even k. One feature of our proof is that it provides a procedure for partitioning a simply-connected orthogonal polygon into at most $\left\lfloor \frac{n}{k+4} \right\rfloor$ polygons of size at most $2k + 6$; this generalizes results in [10], [5] for $k = 0$. The fourth section shows that the lower bound for line guards is tight and that $r(n, h, 2) \leq \lfloor \frac{n}{6} \rfloor$. The last section provides a summary and a discussion of algorithmic aspects and of future directions is given. Due to the space limitation we have to omit several proofs, all details are in [6].

2 Lower bounds on $r(n, h, k)$

In this section, we establish the following lower bounds on $r(n, h, k)$:

$$r(n,h,k) \geq \begin{cases} \left\lfloor \frac{n-2h}{k+4} \right\rfloor & \text{even } k \\ \left\lfloor \frac{3n+(7-3k)h+4}{3k+13} \right\rfloor & k = 1, 3 \\ \left\lfloor \frac{3(n-2h)+4}{3k+13} \right\rfloor & \text{odd } k \geq 5 \end{cases}$$

These bounds are valid only for certain relationships of n/h, and k, which may be thought of as "having enough vertices per hole to make it interesting.". It is known that $r(n, h, 0) = \lfloor \frac{n}{4} \rfloor$ for $k = 0$ [7]. So, let us start with the $\left\lfloor \frac{n-2h}{k+4} \right\rfloor$ bound for even $k \geq 2$ which is valid for $\frac{n}{h} \geq k+6$.

Figure 2 shows examples of infinite polygon classes that establish a lower bound of $\left\lfloor \frac{n}{k+4} \right\rfloor$ for $h = 0$. The figure shows examples (left and right) for $k = 4$ and $k = 6$ which consist of $\frac{n}{k+4}$ spiral arms joined in a row; one guard is needed for each arm. Examples for larger k are made by increasing the number of turns on each spiral arm (one more turn per each increase of two in k). Examples for larger n are made by joining more arms to the polygon. Holes made be added to these examples in the following manner: find a spiral arm that does not contain a hole (here we use the property that $\frac{n}{h} \geq k + 4$), shorten that spiral by one turn, and add a rectangle in its end. This operation increases n by two and h by one, leaving the numerator (of $\left\lfloor \frac{n-2h}{k+4} \right\rfloor$) unchanged, and ensures that each arm still requires its own guard. An examples of this construction is shown in the lower part of Figure 2 for $n = 34, h = 2, k = 6$.

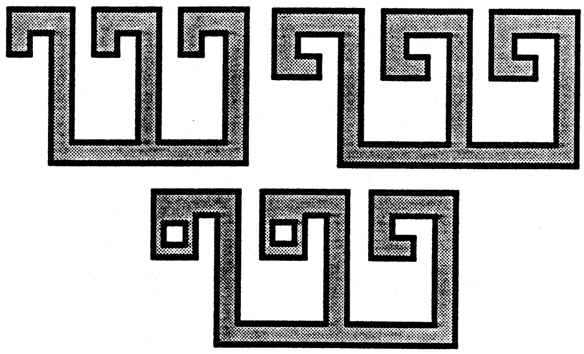

Figure 2: Lower bounds for even k

It remains to show lower bounds for odd k. Note that all both of the bounds that we wish to show (one for $k = 1$ and 3, and another for $k \geq 5$ both simplify to $\left\lfloor \frac{3n+4}{3k+13} \right\rfloor$ for $h = 0$. We first establish this bound, and describe the general construction method for odd k. Let the term *t-pinwheel* denote the $(8t+12,0)$–polygon formed by connecting four spiral arms of t turns in "pinwheel fashion", as illustrated in Figure 3 for $t=3$. In any t–pinwheel the vertices at the end of each spiral arm (one for each arm) form an independent set with respect to paths of length $2t+1$ inside the polygon. Thus, no T_{2t-1}–guard can see two of these vertices.

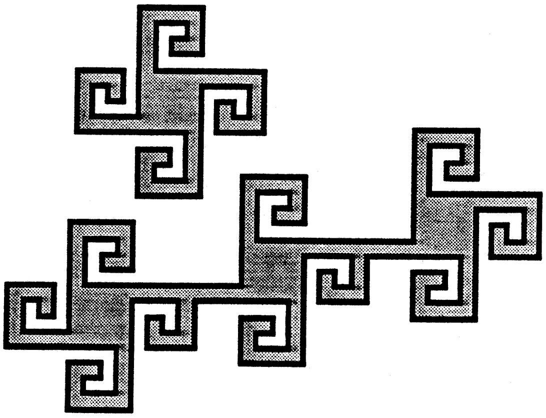

Figure 3: A 3-pinwheel and a 3-growth

The second polygon in Figure 3 illustrares how to construct larger polygons from pinwheels preserving the independence property. We call this operation *grafting* and the resulting polygons *t-growths.* One can easily compute that they give the desired $\left\lfloor \frac{3n+4}{3k+13} \right\rfloor$ lower bound for odd k and $h=0$.

The general $\left\lfloor \frac{3(n-2h)+4}{3k+13} \right\rfloor$ bound can be established by starting with the (holeless) $(\frac{k+1}{2})$–growth and adding holes in the same fashion that we did for the even–k examples.

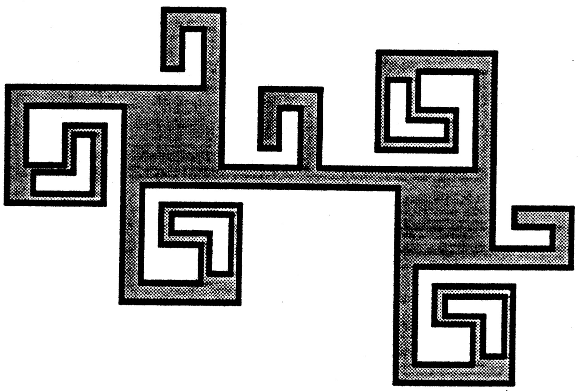

Figure 4: Example for $k=3$

For $k=3$, we wish to show a better lower bound of $\lfloor \frac{3n-2h+4}{22} \rfloor$. We start, as expected, with 2-growths, but to add a hole we *increase* the number of turns on a spiral arm by one, and insert an L-shaped hole that sits inside this turn (see Figure 4 for an example). This process adds 8 vertices and 1 hole ($3\Delta n - 2\Delta h = 22$) but the polygon now requires one extra guard, which bears out the formula. This hole insertion may be carried out as long as $\frac{n}{h} > 19\frac{1}{3}$. For $k=1$, the bound of $\lfloor \frac{3n+4h+4}{16} \rfloor$ is established by starting with 1-growths and adding rectangular holes in the ends of empty spiral arms [15].

3 Upper bound on $r(n,0,k)$

Theorem 1 *Any $(n,0)$–polygon in general position can be partitioned into $\left\lfloor \frac{n}{k+4} \right\rfloor$ simply-connected rectilinear polygons of at most $2k+6$ vertices and thus $r(n,0,k) \leq \left\lfloor \frac{n}{k+4} \right\rfloor$.*

Lemma 2 *Any simply-connected rectilinear polygon of at most $2k+6$ vertices can be covered by one T_k–guard.*

Since this lemma can be proved easily by induction, it is sufficient to give a proof of Theorem 1 for a polygon P with $n \geq 2k+8$ vertices.

We let the term *cut* denote either a chord of the horizontal or vertical rectangular decomposition of P or the L-shaped union of two line segments joining two reflex vertices. We prove Theorem 1 inductively, using cuts to subdivide the polygon P. A cut subdivides P into two rectilinear subpolygons of n_1 and n_2 vertices such that $n_1+n_2=n+2$; we refer to such a cut as a (n_1,n_2)–cut. Such a cut will be called good if $\left\lfloor \frac{n_1}{k+4} \right\rfloor + \left\lfloor \frac{n_2}{k+4} \right\rfloor \leq \left\lfloor \frac{n}{k+4} \right\rfloor$, i.e. if the inductive argument can be applied. We recall once more that if $n < k+4$ then we have to count one for $\left\lfloor \frac{n}{k+4} \right\rfloor$ rather than zero. A straightforward calculation gives the following.

Lemma 3 *Let n, n_1, n_2 be even numbers with $n \geq 2k+8$ and $n_1+n_2=n+2$. An (n_1,n_2)–cut of an $(n,0)$–polygon is good if one of the following conditions holds: (i) $n_1 \leq 2k+6$ and $n_2 \leq 2k+6$*
(ii) $n_1 \geq k+4$ and $n_2 \geq k+4$ and $n_1 \not\equiv 0$ or 1 (mod $k+4$)
(iii) $n_1 \geq k+4$ and $n_2 \geq k+4$ and $n_2 \not\equiv 0$ or 1 (mod $k+4$)
(iv) $n_1 \equiv n_2 \equiv 1$ (mod $k+4$).

If a polygon has an (n_1,n_2)–cut and an (n_1+2, n_2-2)–cut and moreover $n_1 \geq k+4$, $n_2-2 \geq k+4$ then at least one of the cuts is a good cut. If the region between two such cuts is a rectangle they will be called *a pair of consecutive cuts.*

Proof of Theorem 1. As P is an $(n,0)$–polygon, the R–graph $\mathbf{R}(P)$ is a tree with $r=\frac{n-2}{2}$ nodes, and therefore it has a node R such that after deleting it, the size of any connected component is at most $\frac{r}{2}$. In terms of the polygon this means that $deg(R)$ horizontal cuts partition the polygon into $deg(R)+1$ parts: the rectangle R and polygons $P_1, \ldots, P_{deg(R)}$ with $n_1, \ldots, n_{deg(R)}$ vertices. One can easily show that any n_i is at most $\sum_{j\in\{1,\ldots deg(R)\}\setminus\{i\}} n_j + 6 - 2\cdot deg(R)$. Now, we have the three possibilities: R has 2, 3 or 4 neighbors.

Case A: Suppose that $deg(R) = 2$.
We have an (n_1, n_2+2)–cut and an (n_1+2, n_2)–cut. If both n_1 and n_2 are $\geq k+4$ then at least one of the cuts is good. Otherwise, if say $n_1 < k+4$ we get $n_2 \leq n_1+6-2\cdot 2 < k+4+2 \leq 2k+6$. Thus, the (n_1+2, n_2)–cut will be good by Lemma 3 (i).

Case B: Suppose that $deg(R) = 3$ then (by symmetry) we may assume one of the situations in Figure 5. Further it is known that $n_1+n_2+n_3 = n+2$ and $n_i \leq n_j + n_k$ for any permutation (i,j,k). Clearly, we have an (n_1, n_2+n_3)–cut, an (n_2, n_1+n_3)–cut and an (n_3, n_1+n_2)–cut, but, there is also a fourth (n_3+2, n_1+n_2-2)–cut which starts vertically from A down to the horizontal edge thru C or its extension.

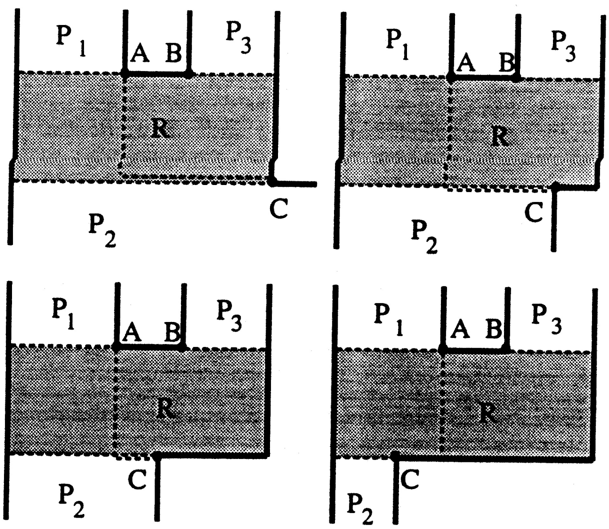

Figure 5: Illustration of Case B

By a careful analysis which makes use of Lemma 3 one can show that for even k one of the four cuts is good. Further this holds also for odd k with the only exception $n_1 \equiv n_2 \equiv 1 \pmod{k+4}, n_3 = k+3$.
Then, let us assume that the vertex C is more left than A (otherwise we have to change their roles and the roles of P_1 and P_2). We define a new cut shooting a vertical ray from C into P_1 until we find a reflex vertex on the right side of the ray which is higher than A (and obtain an L-shaped cut) or until the ray hints a horizontal polygon edge (and obtain a vertical cut). Let P_3' be the subpolygon defined by the new cut which contains P_3. Then it is an easy exercise to prove that either P_3' is of size $k+7$ or one can find a consecutive cut which is inside P_3'.

Case C: If $deg(R) = 4$ this can be reduced to Case B joining together either the two subpolygons on the left or on the right side of R by an L-shaped cut (dependently on which side the sum of the sizes is smaller).
□

4 Upper bounds on $r(n,h,1)$ and $r(n,h,2)$

Theorem 4 *$\lfloor \frac{3n+4h+4}{16} \rfloor$ T_1–guards are always sufficient to cover any rectilinear (n,h)-polygon.*

In fact we prove that these guards can be chosen to be polygon edges or edge extensions.

Let $G_1, \ldots, G_l$ be a family of T_1–guards in an (n,h)–polygon P and D a rectilinear region covered by them (called a district of the guards). Usually, D will be smaller than the maximal possible region covered by $G_1, \ldots, G_l$. Deleting D from P we obtain a number (say c') of connected regions which are $(n_1, h_1), \ldots, (n_{c'}, h_{c'})$–polygons denoted by $P_1, \ldots, P_{c'}$.
The deletion of D will be called a *reduction* if $l + \sum_{i=1}^{c'} \lfloor \frac{3n_i+4h_i+4}{16} \rfloor \leq \lfloor \frac{3n+4h+4}{16} \rfloor$, i.e. if the deletion allows to apply induction. Note, that this definition also makes sense if D is the whole polygon: then we have $c' = 0$, the sum over an empty set is also 0 and we get $l \leq \lfloor \frac{3n+4h+4}{16} \rfloor$.

Using the notations above we define

$$gain(D) \ := \ 3(n-n') + 4(h-h') + 4(1-c')$$

where $n' = \sum_{i=1}^{c'} n_i$, $h' = \sum_{i=1}^{c'} h_i$. Clearly, $gain_P(D) \geq l \cdot 16$ implies that the deletion of D is a reduction.

It will be very helpful to represent $gain(D)$ using the number $r = \frac{n}{2} + h - 1$ of nodes in $\mathbf{R}(P)$. Thus $n = 2(r-h+1)$ and $n' = 2(r'-h'+c')$ where r' is the total number of nodes in the graphs $\mathbf{R}(P_i)$, $1 \leq i \leq c'$ and we get

$$gain(D) = 6(r-r') - 2(h-h') + 10(1-c').$$

The triple $(\delta_r, \delta_h, \delta_c)$, where $\delta_r = r - r'$, $\delta_h = h - h'$, $\delta_c = 1 - c'$, will be called the *type* of D.

Lemma 5 **(Expansion Lemma)** *Let G be a horizontal T_1–guard in a polygon P and D a district of G. Let P_1 be a polygon representing a connected component of $P \setminus D$, and e be a horizontal edge that bounds P_1 from above and is shared between P_1 and D. Let R be the rectangle of P_1 that contains e. Let $\overline{D}$ be the expansion of D by R and all rectangles reachable from R on directed paths in $\mathbf{R}(P_1)$. If the edge e is (orthogonally) visible from G*

(see Figure 6, where G runs across the top of the figure), then $\overline{D}$ is also a district of G and the following holds: $gain(\overline{D}) \geq gain(D) + 6$

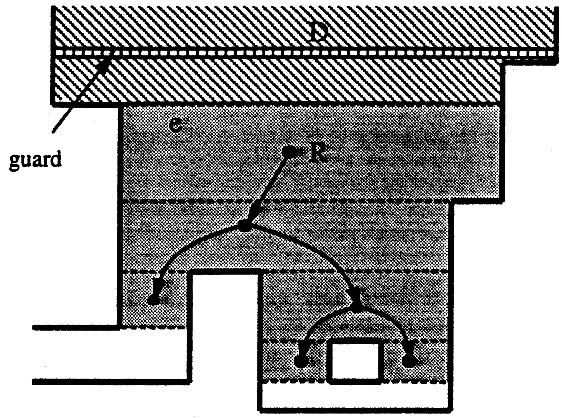

Figure 6: Illustrating Lemma 5

We define the *frame* of $\mathbf{R}$ to be the largest subgraph $\mathbf{F}$ such that for every vertex R in $\mathbf{F}$, $deg_{\mathbf{F}}(R) \geq 2$. Clearly, if we denote by $\mathbf{T}$ the graph $\mathbf{R} \setminus \mathbf{F}$ then any connected component of $\mathbf{T}$ is a tree.

The proof of the theorem now follows from two lemmata which show that each non-trivial polygon is reducible.

Lemma 6 *If $\mathbf{C}$ is a connected component of $\mathbf{T}$ which contains at least three rectangles then for any $R \in \mathbf{C}$ such that $deg_{\mathbf{R}}(R) = 1$ one can find a reduction with R in the reduction district.*

Due to the space limitation we have to omit the proof which consists of a rather long case inspection and several tricky arguments, all details are in [6]. However its difficulty is not surprising because it yields a new proof for simply connected polygons (cf. [1]).

An *extremal hole edge* is a polygon edge e on the boundary of a hole such that

- e connects two reflex vertices and
- in the partition of P induced by extending e in both directions until it hits the boundary, the region containing e is simply-connected.

One can show that if a rectilinear polygon has holes, then it has an extremal hole edge.

Lemma 7 *Let P be a polygon such that any connected component of $\mathbf{T}$ consists of at most two rectangles. W.l.o.g. let e be a horizontal extremal hole edge bounding the hole from above and let $R \in \mathbf{R}$ be the rectangle having e on its boundary. Then there is a reduction such that R or a rectangular part of R is in the district of the reduction.*

Proof: We note that R has two lower neighbors R_l and R_r. If there are also upper neighbors S_1 and S_2 of R then because e is extremal, each of them is either leaf or of degree two and adjacent to some leaf L_1 or L_2. Let $\mathbf{N}$ be the set consisting of all upper neighbors of R and all leaves adjacent to these neighbors. We distinguish three cases:

Case A: Suppose that any rectangle of $\mathbf{N}$ is reachable from R on a directed path in $\mathbf{R}$ (note that this condition holds also if $\mathbf{N}$ is empty).
We place a horizontal guard onto the full extension of e. Clearly, it covers a district D consisting of R and all rectangles of $\mathbf{N}$. Thus, the type of D is $(1+|\mathbf{N}|, 1, 0)$ and its gain is $6 + 6 \cdot |\mathbf{N}| - 2 \geq 4$. Moreover for both R_l and R_r the expansion lemma can be applied, so the expanded district $\overline{D}$ has a gain $\geq 4 + 2 \cdot 6 = 16$.

Case B: Suppose that there is (exactly) one upper neighbor S_1 and an arc $R \leftarrow S_1$.
Placing a horizontal guard onto the upper boundary of S_1 and extending it as far as possible we can cover R and all rectangles of $\mathbf{N}$ and hence we can proceed further as in Case A.

Case C: Suppose that there is (at least) one upper neighbour S_1 adjacent to a leaf L_1 and arcs $R \rightarrow S_1 \leftarrow L_1$.
W.l.o.g. let S_1 be a left neighbor of R. Placing a vertical guard onto the common vertical polygon edge f of R and S_1 and its extension one can cover a district D consisting of L_1, S_1 and that part of R which is bounded by f on the left side and by the extension of the left boundary of R_r on the right side. So after deleting D the remaining part of R forms together with R_r one rectangle in the rectangular decomposition and thus D is of type $(3,1,0)$ and one has $gain(D) = 16$. □

We close this section with stating an upper bound for T_2-guards.

Theorem 8 *For any (n,h)-polygon P we have $r(P,2) \leq \lfloor \frac{n}{6} \rfloor$.*

To prove this theorem one goes along similar lines as in the proof of Theorem 4 where in contrast to the above proof the lemmata for reducing simply connected parts become rather trivial. For reducing holes the existence of extremal edges is also essential. Roughly speaking one can use the second arm of a T_2-guard to cover one rectangle more. The full proof can be found in [6].

5 Algorithmic Aspects and Conclusion

The combinatorial upper bounds proved in this paper have also efficient algorithmic versions.

Theorem 9 *(i) Let k be fixed. Given a rectilinear* $(n, 0)$*-polygon in general position one can partition it in linear time and linear space into at most* $\left\lfloor \frac{n}{k+4} \right\rfloor$ *simply connected rectilinear polygons of at most* $2k+6$ *vertices.*
(ii) In $O(n \log n)$ *time and linear space one finds for a given rectilinear* (n, h)*-polygon a decomposition into at most* $\lfloor \frac{3n+4h+4}{16} \rfloor$ *districts of* T_1*-guards as well as a decomposition into at most* $\lfloor \frac{n}{6} \rfloor$ *districts of* T_2*-guards.*

The details of the algorithms which we have to omit use rather standard methods. In (i) the linear time bound depends both on Chazelle's linear time triangulation of simple polygons [3] and on a modification of the decomposition procedure in Theorem 1 such that it works in a greedy way. In (ii) a sweep–line algorithm can be used to construct the R–graph, see also [4]. That is the only part of the algorithm which needs $O(n \log n)$ time.
Finally we remark that also the partition algorithms in (ii) can be proved to be optimal.

We have found that in the rectilinear world there is a strong difference between odd and even k. Surprisingly, for $k \geq 3$, we have not found lower bounds where increasing h makes polygons require more guards, and we in fact believe that increasing h makes polygons require *less* guards. However, we are unable to establish this, and leave this question unsettled.

We note here that our lower bound constructions give the same bounds even if the usual visibility (rather than rectangle visibility) is used, and the T_k–guards are not rectilinearly embedded; the upper bound arguments (obviously) also hold in this more general situation. The fourth author has previously shown that the even-k upper bound of $r(n, 0, k) \leq \left\lfloor \frac{n}{k+4} \right\rfloor$ holds in this situation [14]; his result is implied by Theorem 1.

There are many questions related to this paper which are yet to be answered. Aside from the usual questions about tight bounds for the generalized guarding problem both for rectilinear and general polygons, we want to mention the following:
(i) What is the lower bound on $r(n, h, k)$ when $\frac{n}{h}$ is small (lots of rectangular holes)?
(ii) Are there lower bound examples that have a different structure but illustrate the same bounds as our constructions? We conjecture that there are no such examples.
(iii) What are the exact bounds for rectilinear polygons with holes expressed as a function only of n and k? (Wessel showed a lower bound of $\lfloor \frac{3n+4}{14} \rfloor$ for $k = 1$ [15].)

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