Link-2-Convex One-Fillable Polygons are Starshaped*

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Abstract

Two points are said to be orthogonally visible if there is a path between them monotone with respect both axes. A polygon $P$ is called orthogonally convex (orthogonally link-2 convex) if every pair of points in $P$ is orthogonally visible (are orthogonally visible from some third point). A polygon $P$ is orthogonally starshaped if there exists a point from which every point in $P$ is orthogonally visible. Motwani, Raghunathan, and Saran provide a polynomial algorithm for the orthogonal star cover problem based on showing that a certain characteristic graph of a polygon is perfect, and that for orthogonal visibility, link-2 convexity implies starshapedness.

In this paper we consider to what extent these techniques can be extended to more general kinds of convexity. In particular we exhibit a non-trivial class of polygons that are starshaped if they are link-2 convex.

1 Introduction

1.1 Definitions

This section contains some required definitions, a review of the some of the relevant previous work on visibility and an outline of this paper. We start with some geometric definitions.

The neighbourhood of any point on the interior of a curve has two well defined sides. A proper crossing of two curves $S_1$ and $S_2$ denotes a (possibly zero length) curve $S_3 \subset \text{int}(S_1) \cap \text{int}(S_2)$ such that we traverse $S_1$ from one endpoint to another, $S_2$ is one side of $S_1$ in the neighbourhood of the first endpoint of $S_3$ encountered, and on the other side of $S_1$ in the neighbourhood the second endpoint of $S_3$ encountered.

A closed polygonal curve $S$ is called weakly simple if any pair of distinct points in $S$ divides $S$ into two polygonal curves that have no proper crossings and the total angle traversed when $S$ is traversed from any point on $S$ is equal to 360 degrees. Like simple polygons, weakly simple closed polygonal curves have a well defined interior and exterior. A weakly simple polygon is defined to be a weakly simple closed polygonal curve, along with the interior of the curve.

In this paper we are interested in those weakly simple subpolygons defined by a chord of a simple polygon. These half polygons consist of a single base edge $l_0r_k$, a set of (possibly zero length) segments $\langle l_0r_0,l_1r_1,\ldots,l_kr_k \rangle$ collinear with $l_0r_k$ and a set of non-intersecting simple polygonal chains $\langle \beta_1 \ldots \beta_k \rangle$ where $\beta_i$ joins $l_i$ to $r_{i-1}$ (see Figure 1). If a half polygon $Q$ contains only one polygonal chain $\beta_i$, then $Q$ is called a hat polygon (see Figure 2). The zero width regions of a hat polygon $Q$ between $l_0r_0$ and the base edge and between $l_1r_1$ and the base edge are called the brim segments of $Q$.

The orientation of a line denotes the smallest angle that the line makes with the positive $x$-
axis. Since lines are undirected, we assume that all line orientations are in the range $[0^\circ, 180^\circ)$. All other objects will have orientations in the range $[0^\circ, 360^\circ)$. Given a set of orientations $O$, we define the span of $O$ to be the smallest angle $\alpha$ such that there exists a wedge of angle $\alpha$ containing every orientation in $O$.

### 1.2 Visibility

Visibility is a central notion in computational geometry. Informally, visibility problems are concerned with whether or not pairs of geometric objects within a set of obstacles can “see” one and other. Recent research has considered generalized visibility or “reachability” problems, where the notion of straight line visibility is generalized to reachability by some sort of constrained path. In this section we review the traditional notion of visibility and some of the generalizations of visibility that researchers have investigated. In particular we describe the notion of restricted orientation visibility that this paper investigates.

Two points $x$ and $y$ in a polygon $P$ are said to be visible if the line segment between them does not intersect the exterior of $P$. A set of points $P$ is said to be convex (link-2 convex) if every pair of points in $P$ is visible (sees some third point in $P$). A set of points $P$ is said to be star-shaped if $P$ contains some point from which all of $P$ is visible.

Orthogonal polygons are commonly studied in computational geometry both because many geometric problems admit simple case analysis based algorithms on orthogonal polygons, and because they arise in many important applications (e.g., in VLSI and image processing). Keil [5], Culberson and Reckhow [4], and Motwani, Raghunathan, and Saran [6, 7] investigated the notion of orthogonal visibility in orthogonal polygons. In many applications not only the boundary but also internal paths (e.g., wires in a chip) are constrained to be chains of orthogonal line segments; in this case it becomes natural to say that two points $x$ and $y$ are orthogonally visible if there is a path between them that is monotone with respect to both axes, since no shorter path is realizable under the constrained geometry. A polygon $P$ is called orthogonally convex if every pair of points in $P$ is orthogonally visible; this is equivalent to requiring that the intersection of $P$ with any horizontal or vertical line be empty or connected. A polygon $P$ is called an orthogonal star if there exists some set of points $K$ in $P$ such that every point in $P$ is orthogonally visible from each point in $K$. Rawlins and Wood [8, 9] generalized orthogonal convexity to the notion of restricted orientation convexity, or $O$-convexity. Let $O$ denote a fixed, but unspecified set of line orientations. A line is called an $O$-line if it has an orientation in $O$. A set $P$ of points is called $O$-concave if it is not $O$-convex. A finite $O$-convex path is called a staircase. Restricted orientation convexity is a generalization of both orthogonal convexity ($O = \{0^\circ, 90^\circ\}$) and the standard notion of convexity ($O = [0^\circ, 180^\circ)$).

**Observation 1** For any orientation $\theta$, there exists an affine transformation that maps horizontal lines to horizontal lines and lines with orientation $\theta$ to vertical lines.

By this observation we may assume without loss of generality that any set $O$ of orientations contains a pair of orthogonal orientations.

Given two points $x$ and $y$ in a polygon, we say that $x$ is $O$-visible to $y$ ($x$ sees $y$) and write $x \sim y$ if there is a staircase between $x$ and $y$ that does not intersect the exterior of the polygon. The following lemma establishes the standard relationship between visibility and convexity:

**Lemma 1** (Rawlins and Wood [9]) If a point set $P$ is connected, then $P$ is $O$-convex if and only if for any pair of points $p$ and $q$ in $P$, $p$ sees $q$.

In this paper we are concerned with $O$-visibility inside polygons.

A point $x$ is called link-$k$ $O$-visible (or just link-$k$ visible) from a point $y$ if there is some path between
$x$ and $y$ (not intersecting the exterior of the polygon) consisting of at most $k$ staircases joined at their endpoints. A set of points $P$ is called link-$k$ \(\mathcal{O}\)-convex (or just link-$k$ convex) if every pair of points in $P$ is link-$k$ \(\mathcal{O}\)-visible.

A set of points $K$ contained in a polygon $P$ is called the \(\mathcal{O}\)-kernel of $P$ if every point in $P$ is \(\mathcal{O}\)-visible from each point in $K$. A polygon is called \(\mathcal{O}\)-starshaped if it contains a non-empty \(\mathcal{O}\)-kernel.

Because we are considering a whole spectrum of different kinds of visibility, it does not suffice to consider classes of polygons, since the visibility properties of a polygon change with type of visibility under consideration. We therefore introduce the notion of a visibility instance, defined to be a pair \((P, \mathcal{O})\) where $P$ is a polygon and $\mathcal{O}$ is a set of orientations.

Motwani et al. [7] have shown that for \(\{0^\circ, 90^\circ\}\)-visibility, link-2 convexity implies starshapedness. They use this, along with some other structural results, to derive a polynomial time algorithm for covering polygons with the minimal number of orthogonally starshaped polygons. In this paper we consider to what extent this result extends to more general sets of orientations $\mathcal{O}$ (i.e. to more general kinds of convexity).

Bose and Toussaint [1] call a polygon $P$ one-fillable if there exists a direction $d$ such that $P$ has only one local maxima with respect to $d$. We will show that one-fillable polygons are starshaped if they are link-2 convex.

The rest of this paper is organized as follows. In Section 2 we present some definitions and basic results on restricted orientation visibility. In Section 3 we give a characterization of when link-2 convexity implies starshapedness. As a corollary of a theorem about \(\mathcal{O}\)-convexity for all $\mathcal{O}$, we establish that one-fillable polygons are starshaped if and only if they are link-2 convex. In Section 4 we present some conclusions and directions for future work.

In this extended abstract we omit many proofs; the reader is referred to [2, 3] for details.

## 2 Dents

In this section we give some definitions and simple lemmas related to restricted orientation visibility. In particular we introduce the notion of a dent which will play a role similar to that of a reflex vertex in the standard notion of visibility.

![Figure 3 The partition of a polygon induced by a pair of oriented chords](image)

**Figure 3** The partition of a polygon induced by a pair of oriented chords

![Figure 4 A dent, and the three subpolygons induced by it.](image)

**Figure 4** A dent, and the three subpolygons induced by it.

An oriented chord is defined to be a pair $\delta = (\gamma, \theta)$ where $\gamma$ is a chord of $P$, and $\theta$ is one of the two orientations perpendicular to $\gamma$. We call an oriented chord $\delta = (\gamma, \theta)$ an \(\mathcal{O}\)-chord if the orientation of $\gamma$ (which is distinct from $\theta$, orientation of $\delta$) belongs to $\mathcal{O}$.

Let $R$ be a weakly simple polygonal region of $P$, and let $\delta = (\gamma, \theta)$ be an oriented chord such that a segment of $\gamma$ is an edge of $R$. We say that $\delta$ faces into (respectively faces out of) $R$ if some ray from $\gamma$ with orientation $\theta$ (respectively $\theta + 180^\circ$) is in $R$ in the neighbourhood of $\gamma$. Each oriented chord divides the polygon into two weakly simple subpolygons; $A(\delta)$ denotes the one that $\delta$ faces into and $B(\delta)$ denotes the one that $\delta$ faces out of (see Figure 3). By convention $\delta$ is included in $A(\delta)$ but not in $B(\delta)$. If $x \in A(\delta)$ then we say that $x$ is \(\mathcal{O}\)-above $\delta$; conversely, if $x \in B(\delta)$, we say that $x$ is \(\mathcal{O}\)-below $\delta$. Where there is no ambiguity, we take "above" to mean \(\mathcal{O}\)-above and "below" to mean \(\mathcal{O}\)-below.

Culberson and Reckhow [4] introduced the term dent to denote an edge of an orthogonal polygon with two reflex endpoints. Let $\tau$ be a reflex vertex or edge such that
1. There exists an O-chord $\delta = (\gamma, \theta)$ such that $\gamma$ is tangent to $\tau$, and

2. Any ray from $\tau$ in the direction $\theta$ is inside $P$ in the neighbourhood of $\tau$.

In this situation we call the ordered pair $D = (\tau, \theta)$ a dent, and call $\delta$ the dent chord of $D$, written $\tilde{D}$. A given reflex vertex may be part of more than one dent, but a given reflex edge may be part of at most one. Given a dent $D = (\tau, \theta)$, $\tau(D)$ denotes $\tau$ and $\theta(D)$ denotes $\theta$. We use the orientation of $D$ to mean the orientation of $\tilde{D}$, $A(D)$ to denote $A(\tilde{D})$, and $B(D)$ to denote $B(\tilde{D})$. Given a set of dents $D$, $D \in \mathcal{D}$ is called a maximal element of $\mathcal{D}$ if

$$(\exists D' \in \mathcal{D}) \ B(D) \subset B(D').$$

We may further subdivide $B(D)$. The dent chord $\tilde{D}$ can be thought of as two disjoint collinear line segments from $\tau(D)$ to the polygon boundary. We define $B_1(D)$ (respectively $B_r(D)$) to be the weakly simple subpolygon induced by $D$ containing the polygon edge clockwise (respectively counterclockwise) from $\tau(D)$ (see Figure 4).

A separating dent for two points $x$ and $y$ is a dent $D$ such that $x \in B_1(D)$ and $y \in B_r(D)$ or vice versa.

Lemma 2 Two points $x$ and $y$ in a polygon $P$ are O-visible if and only if there is no separating dent for $x$ and $y$.

3 Link-2 Convexity and Star-shapedness

To see that any starshaped polygon is link-2 O-convex, we note that any two points in a starshaped polygon see some point in the kernel. A link-2 convex polygon is not necessarily starshaped because every pair of points does not necessarily see the same point $z$. Motwani et al. [7] showed that for $\mathcal{O} = \{0^\circ, 90^\circ\}$ any link-2 O-convex polygon is O-starshaped. By Observation 1, this holds for any $\mathcal{O}$ with $|\mathcal{O}| = 2$. For $|\mathcal{O}| > 2$, link-2 O-convexity is not necessarily equivalent to O-starshapedness. In this section we exhibit a class of visibility instances for which the equivalence of starshapedness and link-2 convexity does hold. This class of visibility instances does not contain all orthogonal visibility instances, but does have an interesting Helly like characterization.

Figure 5 The chord $\gamma_0$ intersects the chord $\gamma_1$.

We say that a set of dents $\mathcal{D}$ covers a polygon $P$ (or $\mathcal{D}$ is a covering set for $P$) if $P = \bigcup_{D \in \mathcal{D}} B(D)$. If $|\mathcal{D}| = 2$, we say that $\mathcal{D}$ is a covering pair for $P$. We say that a dent $D$ covers a point $p$ if $p \in B(D)$.

Lemma 3 A polygon $P$ is O-starshaped if and only if it contains no covering set of dents.

Lemma 4 If $P$ is link-2 convex, then it contains no covering pair of dents.

Let $\mathcal{D}$ be a set of dents. $\Theta(\mathcal{D})$ denotes the set $\{\theta(D_i) \mid D_i \in \mathcal{D}\}$ and the span of $\mathcal{D}$ denotes the span of $\Theta(\mathcal{D})$.

Lemma 5 Let $\mathcal{D}$ be the set of dents in the boundary of a simple polygon $P$ such that the span of $\mathcal{D}$ is at most $180^\circ$. For any dent $D \in \mathcal{D}$, if $(\theta(D) + 180^\circ) \notin \Theta(\mathcal{D})$ then $A(D)$ is a hat polygon.

Lemma 6 Let $P$ be a hat polygon. Let $e = (l, r)$ be an edge of $P$ where $r$ is after $l$ in the counterclockwise traversal of the boundary of $P$. Let $b_0$ and $b_1$ be points in the interior of $e$. Let $\gamma_0 = (b_0, t_0)$ and $\gamma_1 = (b_1, t_1)$ be two chords of $P$ such that either

1. $\gamma_0$ intersects $\gamma_1$ (see Figure 5), or

2. The extensions of $\gamma_0$ and $\gamma_1$ to lines intersect in the half plane defined by $e$ not containing $P$ (see Figure 6).

If $\angle b_0 t_0 r < \angle l b_1 r$, then $t_0$ is encountered before $t_1$ on a counterclockwise walk of the boundary of $P$ from $r$. 

Given two oriented chords \( \delta_0 = (\gamma_0, \phi_0) \) and \( \delta_1 = (\gamma_1, \phi_1) \), we say that \( \delta_0 \) crosses \( \delta_1 \) and write \( \delta_0 \preceq \delta_1 \) if the intersection of the chords \( \gamma_0 \) and \( \gamma_1 \) is a proper crossing. We use \( D_0 \preceq D_1 \) as equivalent notation for \( \delta_0 \preceq \delta_1 \). Let \( l(D) \) (respectively \( r(D) \)) denote the endpoint of \( D \) incident on \( B_l(D) \) (respectively \( B_r(D) \)).

**Lemma 7** Let \( D_0 \) and \( D_1 \) be dents such that \( \{\theta(D_0), \theta(D_1)\} \subseteq [0^\circ, 180^\circ] \) and \( D_0 \preceq D_1 \).

\[
l(D_0) \in B(D_1) \iff \theta(D_1) < \theta(D_0)\]
\[
r(D_0) \in B(D_1) \iff \theta(D_1) > \theta(D_0)\]

**Lemma 8** Let \( D \) be the set of dents in the boundary of a polygon \( P \) with the span of \( D \) at most 180°. If \( D \) contains no covering pair, then \( D \) contains no covering set of dents.

**Proof** Let \( D \) be the set of dents in the boundary of a polygon \( P \). Suppose that the span of \( D \) is at most 180° and \( D \) contains no covering pair. Without loss of generality, suppose the orientations of dents in \( D \) are contained in the (closed) upper half plane induced by a horizontal line. Let \( D' \) be the set of maximal elements of \( D \). If there is a covering set of dents in \( D \) then there is a covering set in \( D' \). Suppose there were a covering set of dents in \( D' \).

We first show the following:

\[
\{ D_0, D_1 \} \subseteq D' \Rightarrow (D_0 \preceq \bar{D}_1) \quad (1)
\]

Let \( D_0 \) and \( D_1 \) be two elements of \( D' \). Since \( D_0 \) and \( D_1 \) are both maximal, it follows that

\[
B(D_0) \not\subset B(D_1) \quad B(D_1) \not\subset B(D_0).\]

Since \( \{ D_0, D_1 \} \) is not a covering pair,

\[
A(D_0) \cap A(D_1) \neq \emptyset.
\]

It follows that \( D_0 \preceq D_1 \).

Suppose the orientation of some \( D \in D' \) were 0° or 180°; it follows the associated dent chord \( \bar{D} \) would be vertical. Let \( p \) be the highest of \( \{ l(D), r(D) \} \). Let \( D_p \) be a dent that covers \( p \). From (1), \( D \preceq D_p \). But this would imply that \( D \) had an orientation strictly between 0° and 180°, which is a contradiction. Thus if there exists such a \( D_p \), there is no covering set for \( D \). We can now assume that

\[
(D \in D') \Rightarrow (0^\circ < \theta(D) < 180^\circ) \quad (2)
\]

Let \( D^* \) be some element of \( D' \). A dent \( D \) is called a right dent if \( 0 \leq \theta(D) < \theta(D^*) \), and a left dent if \( \theta(D^*) < \theta(D) \leq 180 \). From (1) and Lemma 7 we know that \( l(D^*) \) must be covered by a right dent and \( r(D^*) \) must be covered by a left dent. Let \( \beta \) be the path along the polygon boundary from \( l(D^*) \) to \( r(D^*) \) of above \( D^* \). Let \( D_l \) be the left dent in \( D' \) whose chord intersects \( \beta \) closest to \( l(D^*) \). Let \( D_r \) be the right dent in \( D' \) whose chord intersects \( \beta \) closest to \( r(D^*) \) (see Figure 7).

From (1) \( D_l \) and \( D_r \) must both cross \( D^* \). It follows from Lemma 7 that

\[
\bar{D}_l \cap \bar{\beta} = r(D_l) \quad \bar{D}_r \cap \bar{\beta} = l(D_r)
\]

From Lemma 5, \( A(D^*) \) is a hat polygon. Since \( D_r \preceq D_l \) it follows from Lemma 6 that \( r(D_l) \) must be closer to \( r(D^*) \) on \( \beta \) than \( l(D_r) \) is. Let \( D_e \) be some dent that covers \( r(D_l) \). From (1) \( \bar{D}_r \) must intersect \( \bar{D}_l \). Since \( D_e \) covers \( r(D_l) \) and \( \theta(D_e) \) is contained in the upper half-plane, it follows from

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**Figure 6** The extensions of \( \gamma_0 \) and \( \gamma_1 \) to lines intersect in the halfplane not containing \( P \).

**Figure 7** Dents covering the endpoints of a maximal dent chord.
Lemma 7 that $\theta(D_t) < \theta(D_e)$. Since $D_t$ is a left dent,

$$\theta(D^*) < \theta(D_t) < \theta(D_e) < 180^\circ.$$ 

It follows that $D_e$ is a left dent. Since $D_e \gg D_t$ and $\theta(D_t) < \theta(D_e)$ it follows from Lemma 6 that $\tilde{D}_e$ intersects $\beta$ closer to $l(D^*)$ than $\tilde{D}_t$ does. This contradicts our definition of $D_t$, so there is no covering set of dents in $D'$, hence no covering set of dents in $D$.

We can restate the previous lemma in a manner analogous to Helly’s theorem for planar convex sets.

**Corollary 1** Let $D$ be the set of dents in the boundary of a polygon $P$ with the span of $D$ at most $180^\circ$. Let $A$ be the set $\{A(D) \mid D \in D\}$. If every pair of elements of $A$ has a point in common, then $\bigcap_{Q \in A} Q \neq \emptyset$.

We have now established the following theorem:

**Theorem 1** Let $(P, \mathcal{O})$ be a visibility instance with the span of $\mathcal{O}$ be the set of dents in the boundary $P$ at most $180^\circ$. If $P$ is link-2 $\mathcal{O}$-convex then $P$ is $\mathcal{O}$-starshaped.

If we specialize to the case of $\mathcal{O} = [0^\circ, 180^\circ)$, i.e. the standard notions of visibility and convexity, we have the following corollary:

**Corollary 2** Link-2 convex one-fillable polygons are starshaped.

### 4 Conclusions

We have shown that if the span of the orientations of dents in the boundary of a polygon $P$ is at most $180^\circ$ then $\mathcal{O}$-star cover is reducible to link-2 $\mathcal{O}$-convex cover. Let $\mathcal{P}_2$ be defined to be the class of visibility instances $(P, \mathcal{O})$ such that $|\mathcal{O}| = 2$. The results of this paper do not imply those of Motwani et al. [6]; thus we are left with the following open question.

**Question 1** Is there a class $\mathcal{P}_c$ of visibility instances such that $\mathcal{P}_2 \subset \mathcal{P}_c$, and $\forall(P, \mathcal{O}) \in \mathcal{P}_c$ $P$ is $\mathcal{O}$-starshaped if and only if $P$ is link-2 $\mathcal{O}$-convex?

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### References


