Link-2-Convex One-Fillable Polygons are Starshaped*

David Bremner
School of Computer Science
McGill University†

Abstract

Two points are said to be orthogonally visible if there is a path between them monotone with respect both axes. A polygon P is called orthogonally convex (orthogonally link-2 convex) if every pair of points in P is orthogonally visible (are orthogonally visible from some third point). A polygon P is orthogonally starshaped if there exists a point from which every point in P is orthogonally visible. Motwani, Raghunathan, and Saran provide a polynomial algorithm for the orthogonal star cover problem based on showing that a certain characteristic graph of a polygon is perfect, and that for orthogonal visibility, link-2 convexity implies starshapedness.

In this paper we consider to what extent these techniques can be extended to more general kinds of convexity. In particular we exhibit a non-trivial class of polygons that are starshaped if they are link-2 convex.

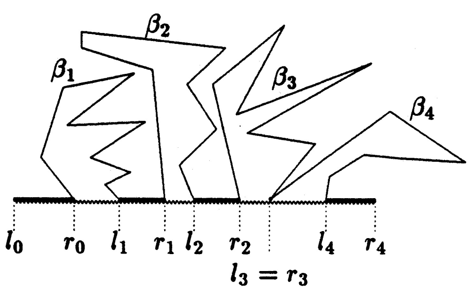
1 Introduction

1.1 Definitions

This section contains some required definitions, a review of the some of the relevant previous work on visibility and an outline of this paper. We start with some geometric definitions.

The neighbourhood of any point on the interior of a curve has two well defined sides. A *proper crossing* of two curves S_1 and S_2 denotes a (possibly zero length) curve $S_3 \subseteq \text{int}(S_1) \cap \text{int}(S_2)$ such that as we traverse S_1 from one endpoint to another, S_2 is one side of S_1 in the neighbourhood

Figure 1 A half polygon.



of the first endpoint of S_3 encountered, and on the other side of S_1 in the neighbourhood the second endpoint of S_3 encountered.

A closed polygonal curve S is called *weakly simple* if any pair of distinct points in S divides S into two polygonal curves that have no proper crossings and the total angle traversed when S is traversed from any point on S is equal to 360 degrees. Like simple polygons, weakly simple closed polygonal curves have a well defined interior and exterior. A *weakly simple polygon* is defined to be a weakly simple closed polygonal curve, along with the interior of the curve.

In this paper we are interested in those weakly simple subpolygons defined by a chord of a simple polygon. These *half polygons* consist of a single base edge $\overline{l_0 r_k}$, a set of (possibly zero length) segments $\langle \overline{l_0 r_0}, \overline{l_1 r_1}, \ldots \overline{l_k r_k} \rangle$ collinear with $\overline{l_0 r_k}$ and a set of non-intersecting simple polygonal chains $\langle \beta_1 \ldots \beta_k \rangle$ where β_i joins l_i to r_{i-1} (see Figure 1). If a half polygon Q contains only one polygonal chain β_i, then Q is called a *hat polygon* (see Figure 2). The zero width regions of a hat polygon Q between $\overline{l_0 r_0}$ and the base edge and between $\overline{l_1 r_1}$ and the base edge are called the *brim segments* of Q.

The orientation of a line denotes the smallest angle that the line makes with the positive x-

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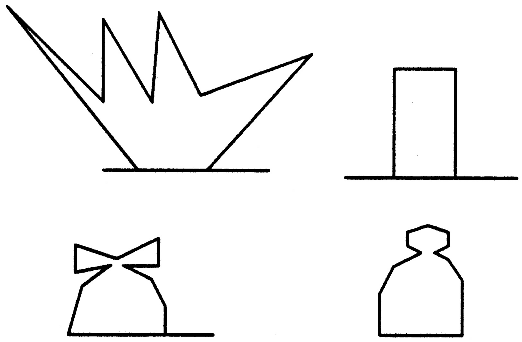
†3480 University St., Montréal, PQ, Canada H3A 2A7
bremner@cs.mcgill.ca

Figure 2 Some example hat polygons.



axis. Since lines are undirected, we assume that all line orientations are in the range $[0°, 180°)$. All other objects will have orientations in the range $[0°, 360°)$. Given a set of orientations $\mathcal{O}$, we define the ***span*** of $\mathcal{O}$ to be the smallest angle α such that there exists a wedge of angle α containing every orientation in $\mathcal{O}$.

1.2 Visibility

Visibility is a central notion in computational geometry. Informally, visibility problems are concerned with whether or not pairs of geometric objects within a set of obstacles can "see" one and other. Recent research has considered generalized visibility or "reachability" problems, where the notion of straight line visibility is generalized to reachability by some sort of constrained path. In this section we review the traditional notion of visibility and some of the generalizations of visibility that researchers have investigated. In particular we describe the notion of *restricted orientation visibility* that this paper investigates.

Two points x and y in a polygon P are said to be *visible* if the line segment between them does not intersect the exterior of P. A set of points P is said to be *convex* (*link-2 convex*) if every pair of points in P is visible (sees some third point in P). A set of points P is said to be *starshaped* if P contains some point from which all of P is visible.

Orthogonal polygons are commonly studied in computational geometry both because many geometric problems admit simple case analysis based algorithms on orthogonal polygons, and because they arise in many important applications (e.g., in VLSI and image processing). Keil [5], Culberson and Reckhow [4], and Motwani, Raghunathan, and Saran [6, 7] investigated the notion of *orthogonal visibility* in orthogonal polygons. In many applications not only the boundary but also internal paths (e.g. wires in a chip) are constrained to be chains of orthogonal line segments; in this case it becomes natural to say that two points x and y are *orthogonally visible* if there is a path between them that is monotone with respect to both axes, since no shorter path is realizable under the constrained geometry. A polygon P is called *orthogonally convex* if every pair of points in P is orthogonally visible; this is equivalent to requiring that the intersection of P with any horizontal or vertical line be empty or connected. A polygon P is called an *orthogonal star* if there exists some set of points K in P such that every point in P is orthogonally visible from each point in K. Rawlins and Wood [8, 9] generalized orthogonal convexity to the notion of *restricted orientation convexity*, or *$\mathcal{O}$-convexity*. Let $\mathcal{O}$ denote a fixed, but unspecified set of line orientations. A line is called an *$\mathcal{O}$-line* if it has an orientation in $\mathcal{O}$. A set P of points is called *$\mathcal{O}$-convex* if the intersection of P with any $\mathcal{O}$-line is either empty or connected. A set P of points is called *$\mathcal{O}$-concave* if it is not $\mathcal{O}$-convex. A finite $\mathcal{O}$-convex path is called a *staircase*. Restricted orientation convexity is a generalization of both orthogonal convexity ($\mathcal{O} = \{0°, 90°\}$) and the standard notion of convexity ($\mathcal{O} = [0°, 180°)$).

Observation 1 *For any orientation θ, there exists an affine transformation that maps horizontal lines to horizontal lines and lines with orientation θ to vertical lines.*

By this observation we may assume without loss of generality that any set $\mathcal{O}$ of orientations contains a pair of orthogonal orientations.

Given two points x and y in a polygon, we say that x is *$\mathcal{O}$-visible* to y (x sees y) and write $x \sim y$ if there is a staircase between x and y that does not intersect the exterior of the polygon. The following lemma establishes the standard relationship between visibility and convexity:

Lemma 1 (Rawlins and Wood [9]) *If a point set P is connected, then P is $\mathcal{O}$-convex if and only if for any pair of points p and q in P, p sees q.*

In this paper we are concerned with $\mathcal{O}$-visibility inside polygons.

A point x is called *link-k* $\mathcal{O}$-visible (or just link-k visible) from a point y if there is some path between

x and y (not intersecting the exterior of the polygon) consisting of at most k staircases joined at their endpoints. A set of points P is called link-k $\mathcal{O}$-convex (or just link-k convex) if every pair of points in P is link-k $\mathcal{O}$-visible.

A set of points K contained in a polygon P is called the ***$\mathcal{O}$-kernel*** of P if every point in P is $\mathcal{O}$-visible from each point in K. A polygon is called ***$\mathcal{O}$-starshaped*** if it contains a non-empty $\mathcal{O}$-kernel.

Because we are considering a whole spectrum of different kinds of visibility, it does not suffice to consider classes of polygons, since the visibility properties of a polygon change with type of visibility under consideration. We therefore introduce the notion of a ***visibility instance***, defined to be a pair $(P, \mathcal{O})$ where P is a polygon and $\mathcal{O}$ is a set of orientations.

Motwani et al. [7] have shown that for $\{0°, 90°\}$-visibility, link-2 convexity implies starshapedness. They use this, along with some other structural results, to derive a polynomial time algorithm for covering polygons with the minimal number of orthogonally starshaped polygons. In this paper we consider to what extent this result extends to more general sets of orientations $\mathcal{O}$ (i.e. to more general kinds of convexity).

Bose and Toussaint [1] call a polygon P *one-fillable* if there exists a direction d such that P has only one local maxima with respect to d. We will show that one-fillable polygons are starshaped if they are link-2 convex.

The rest of this paper is organized as follows. In Section 2 we present some definitions and basic results on restricted orientation visibility. In Section 3 we give a characterization of when link-2 convexity implies starshapedness. As a corollary of a theorem about $\mathcal{O}$-convexity for all $\mathcal{O}$, we establish that one-fillable polygons are starshaped if and only if they are link-2 convex. In Section 4 we present some conclusions and directions for future work.

In this extended abstract we omit many proofs; the reader is referred to [2, 3] for details.

2 Dents

In this section we give some definitions and simple lemmas related to restricted orientation visibility. In particular we introduce the notion of a *dent* which will play a role similar to that of a reflex vertex in the standard notion of visibility.

Figure 3 The partition of a polygon induced by a pair of oriented chords

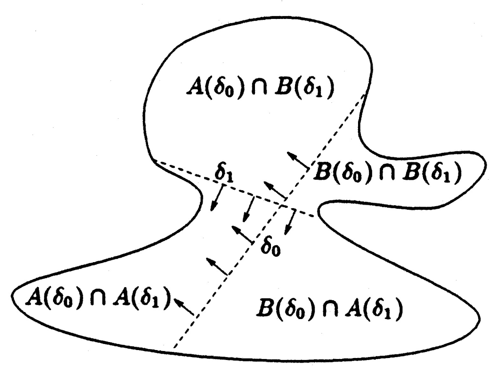
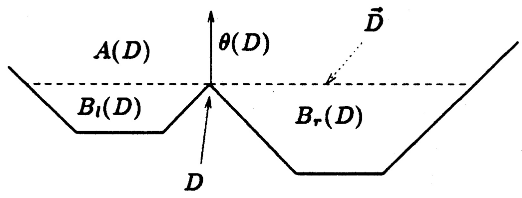


Figure 4 A dent, and the three subpolygons induced by it.



An *oriented chord* is defined to be a pair $\delta = (\gamma, \theta)$ where γ is a chord of P, and θ is one of the two orientations perpendicular to γ. We call an oriented chord $\delta = (\gamma, \theta)$ an ***$\mathcal{O}$-chord*** if the orientation of γ (which is distinct from θ, orientation of δ) belongs to $\mathcal{O}$.

Let R be a weakly simple polygonal region of P, and let $\delta = (\gamma, \theta)$ be an oriented chord such that a segment of γ is an edge of R. We say that δ *faces into* (respectively *faces out of*) R if some ray from γ with orientation θ (respectively $\theta + 180°$) is in R in the neighbourhood of γ. Each oriented chord divides the polygon into two weakly simple subpolygons; $A(\delta)$ denotes the one that δ faces into and $B(\delta)$ denotes the one that δ faces out of (see Figure 3). By convention δ is included in $A(\delta)$ but not in $B(\delta)$. If $x \in A(\delta)$ then we say that x is ***$\mathcal{O}$-above*** δ; conversely, if $x \in B(\delta)$, we say that x is ***$\mathcal{O}$-below*** δ. Where there is no ambiguity, we take "above" to mean $\mathcal{O}$-above and "below" to mean $\mathcal{O}$-below.

Culberson and Reckhow [4] introduced the term *dent* to denote an edge of an orthogonal polygon with two reflex endpoints. Let r be a reflex vertex or edge such that

1. There exists an $\mathcal{O}$-chord $\delta = (\gamma, \theta)$ such that γ is tangent to τ, and
2. Any ray from τ in the direction θ is inside P in the neighbourhood of τ.

In this situation we call the ordered pair $D = (\tau, \theta)$ a *dent*, and call δ the *dent chord* of D, written $\vec{D}$. A given reflex vertex may be part of more than one dent, but a given reflex edge may be part of at most one. Given a dent $D = (\tau, \theta)$, $\tau(D)$ denotes τ and $\theta(D)$ denotes θ. We use the *orientation of* D to mean the orientation of $\vec{D}$, $A(D)$ to denote $A(\vec{D})$, and $B(D)$ to denote $B(\vec{D})$. Given a set of dents $\mathcal{D}$, $D \in \mathcal{D}$ is called a *maximal* element of $\mathcal{D}$ if

$$(\nexists D' \in \mathcal{D})\ B(D) \subset B(D').$$

We may further subdivide $B(D)$. The dent chord $\vec{D}$ can be thought of as two disjoint collinear line segments from $\tau(D)$ to the polygon boundary. We define $B_l(D)$ (respectively $B_r(D)$) to be the weakly simple subpolygon induced by D containing the polygon edge clockwise (respectively counterclockwise) from $\tau(D)$ (see Figure 4).

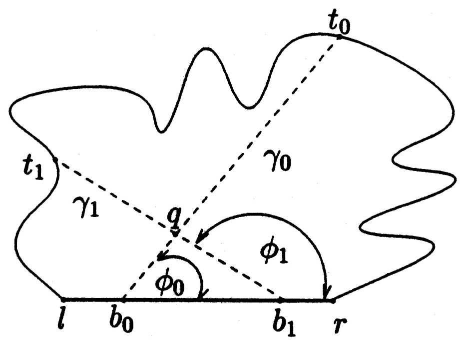
A *separating dent* for two points x and y is a dent D such that $x \in B_l(D)$ and $y \in B_r(D)$ or vice versa.

Lemma 2 *Two points x and y in a polygon P are $\mathcal{O}$-visible if and only if there is no separating dent for x and y.*

3 Link-2 Convexity and Starshapedness

To see that any starshaped polygon is link-2 $\mathcal{O}$-convex, we note that any two points in a starshaped polygon see some point in the kernel. A link-2 convex polygon is not necessarily starshaped because every pair of points does not necessarily see the *same* point z. Motwani et al. [7] showed that for $\mathcal{O} = \{0°, 90°\}$ any link-2 $\mathcal{O}$-convex polygon is $\mathcal{O}$-starshaped. By Observation 1, this holds for any $\mathcal{O}$ with $|\mathcal{O}| = 2$. For $|\mathcal{O}| > 2$, link-2 $\mathcal{O}$-convexity is not necessarily equivalent to $\mathcal{O}$-starshapedness. In this section we exhibit a class of visibility instances for which the equivalence of starshapedness and link-2 convexity does hold. This class of visibility instances does not contain all orthogonal visibility instances, but does have an interesting Helly like characterization.

Figure 5 The chord γ_0 intersects the chord γ_1.



We say that a set of dents $\mathcal{D}$ *covers* a polygon P (or $\mathcal{D}$ is a *covering set* for P) if $P = \bigcup_{D \in \mathcal{D}} B(D)$. If $|\mathcal{D}| = 2$, we say that $\mathcal{D}$ is a *covering pair* for P. We say that a dent D covers a point p if $p \in B(D)$.

Lemma 3 *A polygon P is $\mathcal{O}$-starshaped if and only if it contains no covering set of dents.*

Lemma 4 *If P is link-2 convex, then it contains no covering pair of dents.*

Let $\mathcal{D}$ be a set of dents. $\Theta(\mathcal{D})$ denotes the set $\{\theta(D_i) \mid D_i \in \mathcal{D}\}$ and the *span of* $\mathcal{D}$ denotes the span of $\Theta(\mathcal{D})$.

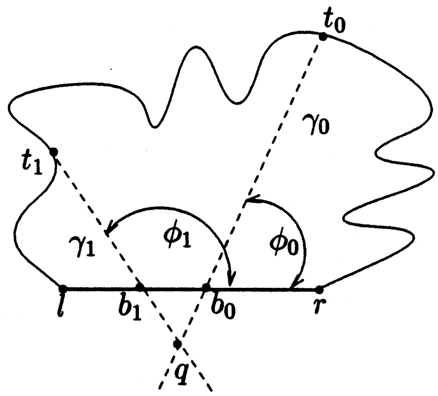
Lemma 5 *Let $\mathcal{D}$ be the set of dents in the boundary of a simple polygon P such that the span of $\mathcal{D}$ is at most 180°. For any dent $D \in \mathcal{D}$, if $(\theta(D) + 180°) \notin \Theta(\mathcal{D})$ then $A(D)$ is a hat polygon.*

Lemma 6 *Let P be a hat polygon. Let $e = (l, r)$ be an edge of P where r is after l in the counterclockwise traversal of the boundary of P. Let b_0 and b_1 be points in the interior of e. Let $\gamma_0 = (b_0, t_0)$ and $\gamma_1 = (b_1, t_1)$ be two chords of P such that either*

1. *γ_0 intersects γ_1 (see Figure 5), or*
2. *The extensions of γ_0 and γ_1 to lines intersect in the half plane defined by e not containing P (see Figure 6).*

If $\angle t_0 b_0 r < \angle t_1 b_1 r$, then t_0 is encountered before t_1 on a counterclockwise walk of the boundary of P from r.

Figure 6 The extensions of γ_0 and γ_1 to lines intersect in the halfplane not containing P.



Given two oriented chords $\delta_0 = (\gamma_0, \phi_0)$ and $\delta_1 = (\gamma_1, \phi_1)$, we say that δ_0 *crosses* δ_1 and write $\delta_0 \bowtie \delta_1$ if the intersection of the chords γ_0 and γ_1 is a proper crossing. We use $D_0 \bowtie D_1$ as equivalent notation for $\vec{D}_0 \bowtie \vec{D}_1$. Let $l(D)$ (respectively $r(D)$) denote the endpoint of $\vec{D}$ incident on $B_l(D)$ (respectively $B_r(D)$).

Lemma 7 *Let D_0 and D_1 be dents such that $\{\theta(D_0), \theta(D_1)\} \subseteq [0°, 180°]$ and $D_0 \bowtie D_1$.*

$$\begin{aligned} l(D_0) \in B(D_1) &\Leftrightarrow \theta(D_1) < \theta(D_0) \\ r(D_0) \in B(D_1) &\Leftrightarrow \theta(D_1) > \theta(D_0) \end{aligned}$$

Lemma 8 *Let $\mathcal{D}$ be the set of dents in the boundary of a polygon P with the span of $\mathcal{D}$ at most 180°. If $\mathcal{D}$ contains no covering pair, then $\mathcal{D}$ contains no covering set of dents.*

Proof Let $\mathcal{D}$ be the set of dents in the boundary of a polygon P. Suppose that the span of $\mathcal{D}$ is at most 180° and $\mathcal{D}$ contains no covering pair. Without loss of generality, suppose the orientations of dents in $\mathcal{D}$ are contained in the (closed) upper half plane induced by a horizontal line. Let $\mathcal{D}'$ be the set of maximal elements of $\mathcal{D}$. If there is a covering set of dents in $\mathcal{D}$ then there is a covering set in $\mathcal{D}'$. Suppose there were a covering set of dents in $\mathcal{D}'$.

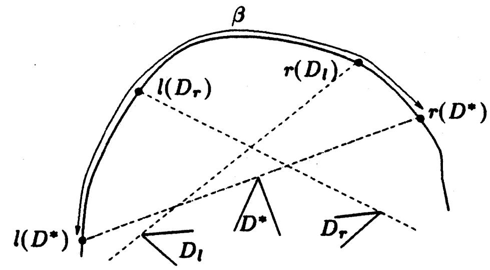
We first show the following.

$$(\{D_0, D_1\} \subseteq \mathcal{D}') \Rightarrow (\vec{D}_0 \bowtie \vec{D}_1) \qquad (1)$$

Let D_0 and D_1 be two elements of $\mathcal{D}'$. Since D_0 and D_1 are both maximal, it follows that

$$\begin{aligned} B(D_0) &\not\subset B(D_1) \\ B(D_1) &\not\subset B(D_0). \end{aligned}$$

Figure 7 Dents covering the endpoints of a maximal dent chord.



Since $\{D_0, D_1\}$ is not a covering pair,

$$A(D_0) \cap A(D_1) \neq \emptyset.$$

It follows that $D_0 \bowtie D_1$.

Suppose the orientation of some $D \in \mathcal{D}'$ were 0° or 180°; it follows the associated dent chord $\vec{D}$ would be vertical. Let p be the highest of $\{l(D), r(D)\}$. Let D_p be a dent that covers p. From (1), $D \bowtie D_p$. But this would imply that D had an orientation strictly between 180° and 360°, which is a contradiction. Thus if there exists such a D_p, there is no covering set for D. We can now assume that

$$(D \in \mathcal{D}') \Rightarrow (0° < \theta(D) < 180°) \qquad (2)$$

Let D^* be some element of $\mathcal{D}'$. A dent D is called a *right* dent if $0 \leq \theta(D) < \theta(D^*)$, and a *left* dent if $\theta(D^*) < \theta(D) \leq 180$. From (1) and Lemma 7 we know that $l(D^*)$ must be covered be a right dent and $r(D^*)$ must be covered by a left dent. Let β be the path along the polygon boundary from $l(D^*)$ to $r(D^*)$ $\mathcal{O}$-above D^*. Let D_l be the left dent in $\mathcal{D}'$ whose chord intersects β closest to $l(D^*)$. Let D_r be the right dent in $\mathcal{D}'$ whose chord intersects β closest to $r(D^*)$ (see Figure 7).

From (1) D_l and D_r must both cross D^*. It follows from Lemma 7 that

$$\begin{aligned} \vec{D}_l \cap \beta &= r(D_l) \\ \vec{D}_r \cap \beta &= l(D_r) \end{aligned}$$

From Lemma 5, $A(D^*)$ is a hat polygon. Since $D_r \bowtie D_l$ it follows from Lemma 6 that $r(D_l)$ must be closer to $r(D^*)$ on β than $l(D_r)$ is. Let D_c be some dent that covers $r(D_l)$. From (1) $\vec{D}_c$ must intersect $\vec{D}_l$. Since D_c covers $r(D_l)$ and $\theta(D_c)$ is contained in the upper half-plane, it follows from

Lemma 7 that $\theta(D_l) < \theta(D_c)$. Since D_l is a left dent,

$$\theta(D^*) < \theta(D_l) < \theta(D_c) < 180°.$$

It follows that D_c is a left dent. Since $D_c \bowtie D_l$ and $\theta(D_l) < \theta(D_c)$ it follows from Lemma 6 that $\vec{D}_c$ intersects β closer to $l(D^*)$ than $\vec{D}_l$ does. This contradicts our definition of D_l, so there is no covering set of dents in $\mathcal{D}'$, hence no covering set of dents in $\mathcal{D}$. ■

We can restate the previous lemma in a manner analogous to Helly's theorem for planar convex sets.

Corollary 1 *Let $\mathcal{D}$ be the set of dents in the boundary of a polygon P with the span of $\mathcal{D}$ at most 180°. Let $\mathcal{A}$ be the set $\{A(D) \mid D \in \mathcal{D}\}$. If every pair of elements of $\mathcal{A}$ has a point in common, then $\bigcap_{Q \in \mathcal{A}} Q \neq \emptyset$.*

We have now established the following theorem:

Theorem 1 *Let $(P, \mathcal{O})$ be a visibility instance with the span of be the set of dents in the boundary P at most 180°. If P is link-2 $\mathcal{O}$-convex then P is $\mathcal{O}$-starshaped.*

If we specialize to the case of $\mathcal{O} = [0°, 180°)$, i.e. the standard notions of visibility and convexity, we have the following corollary:

Corollary 2 *Link-2 convex one-fillable polygons are starshaped.*

4 Conclusions

We have shown that if the span of the orientations of dents in the boundary of a polygon P is at most 180° then $\mathcal{O}$-star cover is reducible to link-2 $\mathcal{O}$-convex cover. Let $\mathcal{P}_2$ be defined to be the class of visibility instances $(P, \mathcal{O})$ such that $|\mathcal{O}| = 2$. The results of this paper do not imply those of Motwani et al. [6]; thus we are left with the following open question.

Question 1 *Is there a class $\mathcal{P}_\pi$ of visibility instances such that $\mathcal{P}_2 \subset \mathcal{P}_\pi$, and $\forall (P, \mathcal{O}) \in \mathcal{P}_\pi$ P is $\mathcal{O}$-starshaped if and only if P is link-2 $\mathcal{O}$-convex?*

5 Acknowledgments

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