Homeomorphic Meshes in $\mathbb{R}^3$

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Abstract

For employing the finite element method for solving partial differential equations, a preliminary step is to decompose the given geometric domain into a mesh. One of the requirements of automatic mesh generation is that the generated mesh should be homeomorphic (that is, topologically equivalent) to the given geometry. We present sufficient conditions that guarantee such a homeomorphism.

Our approach is based on the well-studied octree decomposition [9] of the given domain. The root of the octree is a cube enclosing the domain. A node of the octree is either a leaf cube or has eight equal sized cubes as children. For a leaf cube $C$ and some geometry $X$, let $C_X = C \cap X$ denote the restricted cube of $C$ with respect to $X$. The sufficient conditions require that the restricted cubes of all leaves of the octree with respect to the domain and with respect to the boundary of the domain be topological closed balls. We describe how to construct appropriate edges, triangles and tetrahedra of a mesh so that the mesh is homeomorphic to the domain. The vertex set of the mesh constructed by our method is the set of centroids of all non-empty restricted cubes.

Keywords. Homeomorphism, mesh-generation, octrees, simplicial complexes, triangulations.

1 Introduction

One approach to solve partial differential equations numerically is to use the finite element method. As a preliminary step, this method involves the generation of discretizations of the geometric domain over which the partial differential equation is being solved, see e.g. [13]. The process of discretizing the domain is called mesh generation. Surveys of mesh generation methods can be found in [5, 12]. In this paper, we study octree based meshes and describe conditions which guarantee a homeomorphism between the generated mesh and the given geometric domain in $\mathbb{R}^3$. Below, we introduce some definitions from topology [6, 8].

Basic definitions. A simplex $\sigma_T$ is the convex hull of an affinely independent point set $T$. Its dimension is $k = \dim \sigma_T = \text{card } T - 1$, where $\text{card } T$ denotes the cardinality of $T$. $\sigma_T$ is also referred to as a $k$-simplex. For $d = 0, 1, 2, 3$, a $d$-simplex is respectively called a vertex, an edge, a triangle and a tetrahedron. If $U \subseteq T$, then $\sigma_U$ is called a face of $\sigma_T$. A simplicial complex $K$ is a finite collection of simplices such that the following properties hold.

(i) If $\sigma_U \in K$ and $V \subseteq U$, then $\sigma_V \in K$.
(ii) If $\sigma_U, \sigma_V \in K$, then $\sigma_U \cap \sigma_V \in K$.

The first property says that all faces of $\sigma_U \in K$, including the empty set, are in $K$ and the second property says that two simplices in $K$ intersect in a common face. The vertex set of $K$ is $\text{vert } K = \bigcup_{\sigma_T \in K} T$, the dimension of $K$ is $\dim K = \max_{\sigma_T \in K} \dim \sigma_T$, and the underlying space of $K$ is $|K| = \bigcup_{\sigma_T \in K} \sigma_T$. A subcomplex of $K$ is a simplicial complex $L \subseteq K$.

A geometric triangulation of a point set $S \subseteq \mathbb{R}^d$ is a simplicial complex $K$ with $\text{vert } K \subseteq S$ and $|K| = \text{conv } S$. The popular Delaunay triangulation [2, 3, 7] of a point set is an example of a geometric triangulation. In this paper we shall be interested in a topological triangulation of a topological space in $\mathbb{R}^3$. A topological triangulation of a topological space $X$ is a simplicial complex homeomorphic to $X$. A bijection $f : X \to Y$ is a homeomorphism between $X$ and $Y$ if $f$ and $f^{-1}$ are continuous. If such an $f$ exists, then $X$ and $Y$ are homeomorphic or are homeomorphs, writ-

\[ \text{conv } S \text{ denotes the convex hull of } S. \]
ten $X \cong Y$. A simplicial complex $K$ is homeomorphic to $X$ if the underlying space of $K$ is homeomorphic to $X$.

Background. The meshes that we consider in this paper are simplicial complexes. Schroeder and Shephard [9] present a method for generating meshes automatically. Though one of the issues addressed by their paper is topological equivalence between the mesh and the domain, the paper does not guarantee a topological equivalence. Chew [1] describes a method to construct a triangulation of a surface imbedded in $\mathbb{R}^3$. Extending the idea in [1], Edelsbrunner and Shah [4] present sufficient conditions which guarantee topological equivalence between a given geometric domain $X$ and a simplicial complex meshing $X$. The conditions are formulated in terms of restricted Voronoi cells, which are intersections of Voronoi cells [14] and $X$, see [4, 10] for details. The simplicial complex obtained by their method is a subcomplex of the Delaunay triangulation of a carefully chosen point set (vertex set). It is not clear how to choose this vertex set. In this paper we extend the ideas in [4] to octree based triangulations. This approach implicitly addresses the vertex selection issue.

Outline. In section 2 we define octree triangulations $O_X$ restricted by $X$. In sections 3 and 4 we present conditions that guarantee a homeomorphism between $|O_X|$ and $X$. We conclude the paper with some remarks in section 5.

Remark. The intent of this paper is to present a possibly useful technique for homeomorphic mesh generation without going into tedious technical details. Some proofs are either omitted or only sketched briefly. Complete proofs can be found in [11] along with extensions to 3-manifolds in $\mathbb{R}^4$.

2 Restricted Octree Triangulations

An octree [9] is a geometric division of $\mathbb{R}^3$ into a finite tree of 3-cubes$^2$. Each 3-cube is either a leaf of the tree or an internal node. If a 3-cube is an internal node of the tree, then it is split into 8 equal volume children. Two 3-cubes are neighbors if one contains a face$^3$ of another. An edge $e$ of a 3-cube $C$ is split if any neighbor $C'$ of $C$ incident to $e$ is split. A corner of a 3-cube is one of its 8 vertices. An octree is balanced if an edge of any unsplit cube contains at most one corner in its interior. If a $k$-face, $k > 0$, of one 3-cube of a balanced octree is contained in another, then the two cubes have equal edge length, or the ratio of the larger edge length to the smaller is $2 : 1$. If $k = 0$, then the ratio can also possibly be $4 : 1$. A balanced octree is strictly balanced if the $4 : 1$ ratio is disallowed. Throughout this paper we shall be concerned only with strictly balanced octrees. We shall further assume that the 3-cubes are rectilinear, that is each facet$^4$ of a 3-cube is orthogonal to some unit vector $e_i$, $1 \leq i \leq 3$.

Let $p$ be the centroid of some 3-cube $C_p$. For a given octree $\tau$, let $S = S_\tau$ denote the set consisting of all centroids $p$ such that $C_p$ is a leaf cube of $\tau$. Let $X \subset \mathbb{R}^3$ be a topological space of interest. For $p \in S$, let $C_{p,X} = C_p \cap X$ be the cube $C_p$ restricted by $X$. For $T \subseteq S$, let

$$C_T = \bigcap_{p \in T} C_p,$$

and

$$C_{T,X} = \bigcap_{p \in T} C_{p,X} = C_T \cap X.$$

For $T \subseteq S$, define $\mu_T$ to be the maximum subset $T'$ of $S$ such that $C_{T'} = C_T$. Observe that $C_T$ is a (possibly empty) cube. Define $\ell_T$ so that $C_T$ is a $(4 - \ell_T)$-cube. Note that $1 \leq \ell_T \leq 4$. Before we define the octree triangulation $O_X$ restricted by $X$, we need the following lemma.

Lemma 2.1 Let $\tau$ be a strictly balanced octree and let $S = S_\tau$. Let $T \subseteq S$ be such that $C_T \neq \emptyset$. Then,

(i) $\text{conv } \mu_T \subseteq \bigcup_{p \in \mu_T} C_p$,

(ii) if $(\mu_T)_\perp$ denotes the projection of $\mu_T$ on the $(\ell_T - 1)$-flat orthogonal to aff $C_T$, then the points in $(\mu_T)_\perp$ are in convex position.

Proof. (i) A facet $f$ of a 3-cube $C_p$, $p \in \mu_T$, is free if it is contained in the boundary of $\bigcup_{p \in \mu_T} C_p$. See figure 2.1 for an illustration in $\mathbb{R}^2$. Let $f_p$ be a

$^2$Later, we will use 2-cube to denote a square, 1-cube to denote an edge and a 0-cube to denote a vertex.

$^3$A face of a 3-cube is any of its vertex, edge or bounding square.

$^4$A facet of a $d$-cube is any of its $(d - 1)$-faces.

$^5$aff $Y$ denotes the affine hull of $Y$. 

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free facet for some $p \in \mu_T$. The closed half-space of $\text{aff}_f P$ containing $p$ also contains all points in $\mu_T$ because $\tau$ is strictly balanced. The intersection of all such half-spaces for all free facets is the residual cube $P$, a convex polytope contained in $\bigcup_{p \in \mu_T} C_p$. Since $\mu_T \subseteq P$, it follows that $\text{conv} \mu_T \subseteq P \subseteq \bigcup_{p \in \mu_T} C_p$.

(ii) Let $q$ be any point in $(\mu_T)_\perp$. Without loss of generality assume that $a)$ the point $h \cap \text{aff} C_T$ is the origin, (b) $h$ is identified with $\mathbb{R}^k$, $k \leq 3$, and (c) $q$ is the point $(1, 1, \ldots, 1) \in \mathbb{R}^k$. It can be verified by enumerating all possible configurations of $\mu_T$ under the restriction that $\tau$ be strictly balanced that the halfspace $H : (q, x) < (q, q) \subseteq h$ contains $(\mu_T)_\perp \setminus \{q\}$. We omit the details.

Assume that the points in $S = S_T$ are indexed so that if $p \in S$ has an index smaller that $q \in S$, then the volume of $C_p$ is no greater than that of $C_q$. Now we are ready to construct $O_X = O_{S,T}$. For every subset $T \subseteq S$ such that $O_{T,T} \neq \emptyset$, we construct a simplicial complex $O(T)$ by induction on $\ell_T$. For $\ell_T = 1$, define $O(T) = \{\sigma_T\}$. For $\ell_T = 2$, define $O(T) = \{\sigma_T\} \cup T$. For $\ell_T = 3$, card $\mu_T = 3$ or card $\mu_T = 4$. If card $\mu_T = 3$, define $O(T)$ to be the simplicial complex consisting of the triangle formed by the three points along with its edges and vertices. Otherwise, let $\mu_T = \{a, b, c, d\}$ such that $\ell_{\{a,b,c\}} = \ell_{\{b,c,d\}} = \ell_{\{c,d,a\}} = 2$. Let $a$ be the point with the smallest index among the points in $\mu_T$. Define

$$O(T) = \{\sigma_{\{a,b,c\}}, \sigma_{\{a,b,d\}}\} \cup \bigcup_{T' \subseteq \mu_T, \ell_{T'} = 2} O(T').$$

By Lemma 2.1(ii), the points in the projection of $T$ into a plane orthogonal to aff $C_T \tau$ are in convex position and so it follows that $O(T)$ must be a simplicial complex. Now let $\ell_T = 4$. Let $p$ be the point in $\mu_T$ with the smallest index. Let $\Delta_T = \bigcup_{T' \subseteq \mu_T, \ell_{T'} = 3} O(T')$. Define $O(T) = \Delta_T \cup \{\text{conv} (\sigma \cup \{p\}) | \sigma \in \Delta_T \text{ does not contain } p\}$. It is shown in [11] that $O(T)$ is a simplicial complex. Finally define

$$O_X = O_{S,T} = \bigcup\limits_{C_T \neq \emptyset} O(T).$$

Lemma 2.1(i) guarantees that $O_X$ is a simplicial complex. $O_X$ is the octree simplicial complex of $S$ restricted by $X$. Note that $O_X$ is uniquely defined for $S$ once we assume a specific indexing of its points.

Intuitively, we construct $O(T)$ by inductively obtaining its triangulated “boundary” and then triangulating the “interior”. For example, if $\ell_T = 4$ and card $\mu_T = 8$, then a possible collection of points for $\mu_T$ is $\{(1, 1, 1), (2, -2, 2), (2, -2, 2), (2, 2, 2)\} \cup \{(1, 1, -1), (2, -2, -2), (-2, 2, 2), (-2, -2, -2)\}$ with $\{(0, 0, 0)\}$ being the intersection of all 8 cubes. Let $T'$ be the set of points with positive 3rd coordinate ($\ell_{T'} = 3$). See figure 2.2. If the smallest indexed point is, say, $p = (1, 1, -1)$, then $O(T')$ contains simplices conv $\{p, a, b, c\}$ and conv $\{p, a, b, d\}$, among others.

See figure 2.3 for an illustration of restricted quad tree triangulations. (A quad tree is a 2-dimensional counterpart of an octree.) Observe that $O_X$ and $O_{X_1}$ are homeomorphic to $X_1$ and $X_2$ respectively, while $O_{X_3}$ is not homeomorphic to $X_3$.

3 Homeomorphic Triangulation for Compact Manifolds

In this section we shall state the conditions under which the restricted octree triangulation is a topological triangulation of $X$. These conditions apply only
when $X$ is a compact $m$-manifold, $0 \leq m < 4$. First, we introduce some terminology.

**Topological balls and manifolds.** Homeomorphs of open, half-open, and closed balls of various dimensions play an important role in the forthcoming discussion. For $k \geq 0$, let $o$ be the origin of $\mathbb{R}^k$ and define

\[
\begin{align*}
H^k &= \{ z = (\xi_1, \ldots, \xi_k) \in \mathbb{R}^k \mid \xi_k \geq 0 \}, \\
B^k &= \{ z \in \mathbb{R}^k \mid |zo| \leq 1 \}, \text{ and} \\
S^{k-1} &= \{ z \in \mathbb{R}^k \mid |zo| = 1 \}.
\end{align*}
\]

For convenience, we define $\mathbb{R}^k = H^k = B^k = S^k = \emptyset$ if $k < 0$. An open $k$-ball is a homeomorph of $\mathbb{R}^k$, a half-open $k$-ball is a homeomorph of $H^k$, a closed $k$-ball is a homeomorph of $B^k$, and a $(k-1)$-sphere is a homeomorph of $S^{k-1}$. For $k \geq 1$ these are disjoint classes of spaces, that is, open balls, half-open balls, closed balls, and spheres are pairwise non-homeomorphic. This is not true for $k = 0$: open, half-open, and closed 0-balls are points, and a 0-sphere is a pair of points.

$X \subseteq \mathbb{R}^3$ is a $k$-manifold without boundary if each $x \in X$ has an open $k$-ball as a neighborhood in $X$. $X \subseteq \mathbb{R}^3$ is a $k$-manifold with boundary if each $x \in X$ has an open or half-open $k$-ball as a neighborhood in $X$. The set of points without open $k$-ball neighborhood forms the boundary, $\text{bd } X$, of $X$. Note that the boundary of a half-open $k$-ball is an open $(k-1)$-ball, which is therefore without boundary. The interior of a manifold $X$ is $\text{int } X = X - \text{bd } X$; it is the set of points with open $k$-ball neighborhoods. A manifold $X \subseteq \mathbb{R}^3$ is compact if it is closed and bounded. A manifold $Y \subseteq X$ is a submanifold of $X$.

**Generic intersection and closed ball properties.** Let $P \subseteq \mathbb{R}^3$ be a convex polyhedron of dimension $\ell$ and let $X \subseteq \mathbb{R}^3$ be an $m$-manifold without boundary. We say that $P$ intersects $X$ generically if $X \cap P = \emptyset$ or $X \cap P$ is an $(m+\ell-3)$-manifold and $X \cap \text{int } P = \text{int } X \cap P$. If $X$ has a non-empty boundary, then $P$ intersects $X$ generically if $P$ intersects $X$ and $\text{bd } X$ generically.

Let $X \subseteq \mathbb{R}^3$ be a compact $m$-manifold, with or without boundary. Let $\tau$ be a strictly balanced tree decomposing $X$ and let $S = S_{\tau}$. $S$ has the generic intersection property for $X$ if for every subset $T \subseteq S$, $C_T$ intersects $X$ generically. $S$ has the closed ball property for $X$ if for every $T \subseteq S$ with $\ell = m+1 - \ell_T$, the following two conditions hold.

(i) $C_{T,X}$ is either empty or a closed $\ell$-ball, and

(ii) $C_{T,\text{bd } X}$ is either empty or a closed $(\ell-1)$-ball.

Remark. The generic intersection and closed ball properties were first defined for restricted Voronoi cells in [4]. See [4] for a related discussion on non-degeneracy.

**Theorem 3.1** Let $X \subseteq \mathbb{R}^3$ be a compact $m$-manifold. Let $\tau$ be a strictly balanced octree such that $S = S_{\tau}$.
has the generic intersection property and the closed ball property for \( X \). Then \( X \approx |O_{S,X}| \).

To prove the theorem, we shall need the following lemma.

**Lemma 3.2** Let \( X, m, \tau \) and \( S \) be as in Theorem 3.1. Define \( C_i = \{ C_{T,X} \mid C_{T,X} \text{ is an } i \text{-ball} \} \), \( 0 \leq i \leq m \). There exist simplicial complexes \( K_i \) and homeomorphisms \( h_i : \bigcup C_i \rightarrow \bigcup K_i = |\Omega(T)| \), so that \( K_{i-1} \subseteq K_i \) and \( h_i \) agrees with \( n_{i-1} \) on \( \bigcup C_{i-1} \). Furthermore there exists a bijection \( \beta : \text{vert } K_m \rightarrow \bigcup_{0 \leq i \leq m} C_i \) so that \( \sigma_T \in K_m \) iff the elements of \( \beta(T) \) can be arranged to form a linear chain under the inclusion relation. \( \square \)

Remark. A generalization of the above theorem to compact 3-manifolds in \( \mathbb{R}^d \) can be found in [11].

4 Extension to General Topological Spaces

The previous section described sufficient conditions for \( O_X \) to be homeomorphic to \( X \) when \( X \) was a manifold. In engineering practice, one encounters geometries which are not necessarily manifolds. In this section we extend the sufficient conditions to cover more general geometries. To present the conditions succinctly, we use finite regular CW complexes. Intuitively, the gist of the conditions is that \( X \) should be decomposed into balls appropriately. See also [4, 10].

A closed ball is called a cell, or a \( k \)-cell if its dimension is \( k \). A finite collection of non-empty cells, \( C \), is a regular CW complex if the cells have pairwise disjoint interiors, and the boundary of each cell is the union of other cells in \( C \). Let \( X \subseteq \mathbb{R}^3 \) be a topological space, let \( \tau \) be a strictly balanced octree decomposing \( X \) and set \( S = S_\tau \). \( S \) has the extended closed ball property for \( X \) if there is a regular CW complex \( C \), with \( X = \bigcup C \), that satisfies the following properties for every \( T \subseteq S \) with \( C_{T,X} \neq \emptyset \):

(i) there is a regular CW complex \( C_T \subseteq C \) so that \( C_{T,X} = \bigcup C_T \),

(ii) the set \( C_T^o = \{ \gamma \in C \mid \text{int } \gamma \subseteq \text{int } C_T \} \) contains a unique cell, \( \eta_T \), so that \( \eta_T \subseteq \gamma \) for every \( \gamma \in C_T^o \),

(iii) if \( \eta_T \) is a \( j \)-cell then \( \eta_T \cap \text{bd } C_T \) is a \( (j-1) \)-sphere, and

(iv) for each integer \( k \) and each \( k \)-cell \( \gamma \in C_T^o - \{ \eta_T \} \), \( \gamma \cap \text{bd } C_T \) is a closed \( (k-1) \)-ball.

Furthermore, \( S \) has the extended generic intersection property for \( X \) if for every \( T \subseteq S \) and every \( \gamma \in C_T-C_T^o \), \( A \text{ polytope is the underlying space of a simplicial complex. A star polytope is a polytope } P \text{ for which there exists a point } p \in P \text{ such that for every point } q \in P, \text{ the edge } pq \text{ is contained in } P. \text{ The set of all such points } p \text{ is called the kernel of } P. \)
there is a $\delta \in C_T^\alpha$ so that $\gamma \subseteq \delta$.

For sake of brevity, we omit a discussion of the above conditions and state the following theorem.

**Theorem 4.1** Let $X \subseteq \mathbb{R}^3$ be a topological space. Let $\tau$ be a strictly balanced octree such that $S = S_\tau$ has the extended generic intersection property and the extended closed ball property for $X$. Then $X \approx |O_5, x|$.

## 5 Concluding Remarks

We have discussed octree triangulations restricted by a topological space $X$ and presented sufficient conditions that guarantee a homeomorphism between the triangulation and the space. Our method is based on decomposing $X$ by cubes, requiring the intersections of the cubes and $X$ to be balls and then appropriately connecting the centroids of the cubes to obtain the restricted octree triangulation. Homeomorphic triangulations were studied in [4]. The octree decomposition of the space $X$ studied in this paper gives an automatic way of generating the vertex set of the triangulation which was a sore point of the method in [4]. It would be interesting to see how this approach performs (the number of simplices generated, the quality of simplices, etc.) on geometries generated by solid modelers. This would require verification of the conditions in sections 3 and 4, which in turn might put constraints on the functional description of the geometry of the domain.

## References


