Regular Polygons are Most Tolerant

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1 Introduction

Suppose we are asked to draw the "most convex" n-gon within[1] a given circle. By most convex we mean that the distance we can allow the vertices to move without ruining convexity is maximum. What n-gon do we choose? Our intuition suggests that the regular n-gon whose circumscribed circle is the given circle is the best choice. This paper shows that our intuition is correct.

Abellanas et al. pose this problem as one of a host of problems which involve the *tolerance* of a geometric object [2]. The tolerance of an object measures its ability to withstand perturbation and maintain some property. Perturbation has been studied as a method for handling degeneracies in an algorithm's input or as a way to quantify and control computational errors [4, 5, 6]. We take the idea of perturbation from this realm of algorithmic robustness and use it to define a property — the tolerance — of a particular geometric object.

Our definition of tolerance is based on the notion of *Epsilon-Predicates* from [6]. Let $\mathcal{O}$ be a set of *objects* with a distance metric $\delta : \mathcal{O} \times \mathcal{O} \to \Re$. Let $\mathcal{P}$ be a predicate defined on $\mathcal{O}$. For $P \in \mathcal{O}$, define

$$\mathcal{P}\text{-}tolerance(P) = \sup\{\epsilon \mid \forall Q \in \mathcal{O} \text{ if } \delta(P,Q) \leq \epsilon \text{ then } \mathcal{P}(Q)\}$$

This definition is a restatement of the definition of tolerance given in [2, 3, 1].

As an example of the preceding definition, we rephrase (and make precise) the problem of finding the most convex n-gon within a given finite circle C. Let $\mathcal{O}$ be the set of n-gons, each specified by an n-tuple of vertices (points in $\Re^2$). Define the distance metric δ to be

$$\delta(P,Q) = \delta((p_1,\ldots,p_n),(q_1,\ldots,q_n)) = \max_{i\in\{1,\ldots,n\}} d(p_i,q_i)$$

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[1] An n-gon is within a circle C if the vertices of the n-gon lie on the boundary or in the interior of C.

where d is the Euclidean distance function for points. Let $\mathcal{P}(Q)$ be the predicate "Q is convex". In this paper, when we refer to the tolerance of an n-gon P we mean $\mathcal{P}$-tolerance(P) with the above definitions.

The problem of finding the most convex n-gon within C is the problem of finding

$$\arg\max_{P \in \mathcal{O} \text{ within } C} \mathcal{P}\textit{-tolerance}(P)$$

Rather than solving this problem directly, we look at the related problem of finding the n-gon with tolerance ϵ that has minimum perimeter. Solving the related problem will allow us to solve the original problem quite simply.

2 Perimeter Minimization

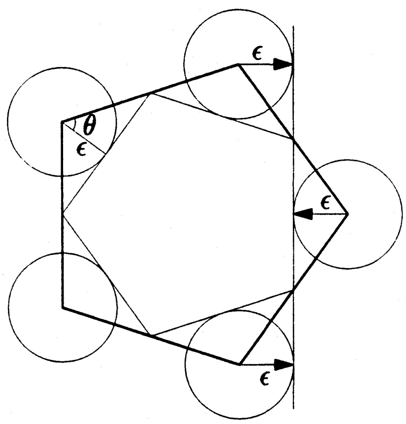

Figure 1: A regular pentagon with tolerance ϵ.

We start by calculating the perimeter of a regular n-gon with tolerance ϵ. Let $\theta = (n-2)\pi/2n$ be half of the interior angle of the regular n-gon. The perimeter of a regular n-gon with tolerance ϵ is

$$2n\frac{\epsilon}{\cos\theta} = \frac{2n\epsilon}{\cos((n-2)\pi/2n)}$$

Notice that the perimeter is proportional to the tolerance for fixed n. In addition, except for triangles which have infinite tolerance, the tolerance of an n-gon is limited by the distance required to make three consecutive vertices on the n-gon co-linear.

THEOREM 1 *For $n > 3$, the regular n-gon with tolerance ϵ has the unique smallest perimeter of any convex n-gon with tolerance ϵ.*

PROOF. Let P be a convex n-gon with tolerance ϵ and vertices $v_1, \ldots, v_n$ in clockwise order. The idea of the proof is to lower bound the perimeter of P as a function of the interior angles of P and its tolerance ϵ, and then show that the regular n-gon uniquely achieves this lower bound.

Let m_i be the midpoint of the edge $\overline{v_i v_{i+1}}$. To avoid an overabundance of notation, we adopt the convention that subscript $n+1 \equiv 1$ and subscript $0 \equiv n$.

We claim that, for all $i \in \{1, \ldots, n\}$, the distance from v_i to the line ℓ_i through m_{i-1} and m_i is at least ϵ. On one side of ℓ_i lies v_i and on the other side lies v_{i-1} and v_{i+1}. Let u_i be the point on ℓ_i closest to v_i, and let ϵ_i be the distance between u_i and v_i.

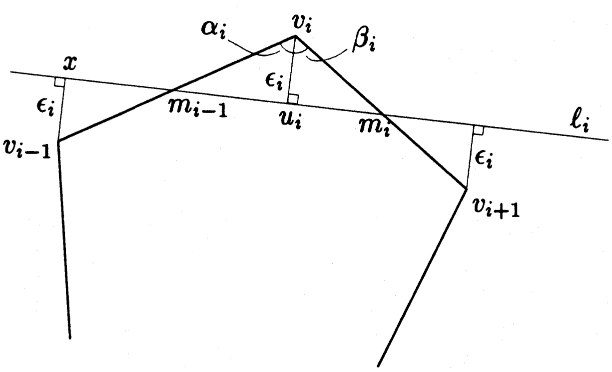

Figure 2: Every vertex v_i is $\epsilon_i \geq \epsilon$ from ℓ_i.

Let x be the point on ℓ_i closest to v_{i-1}. Triangles v_{i-1}, x, m_{i-1} and v_i, u_i, m_{i-1} are congruent, thus v_{i-1} is distance ϵ_i from ℓ_i. Similarly, v_{i+1} is distance ϵ_i from ℓ_i. By moving v_{i-1}, v_i, and v_{i+1} each a distance ϵ_i, we can make these three vertices co-linear. Thus $\epsilon \leq \epsilon_i$ for $i \in \{1, \ldots, n\}$.

Let α_i be the angle formed by v_{i-1}, v_i, u_i, $-\pi/2 < \alpha_i < \pi/2$. Let β_i be the angle formed by u_i, v_i, v_{i+1}, $-\pi/2 < \beta_i < \pi/2$. The sum $\alpha_i + \beta_i$ is the interior angle at vertex v_i. Thus $\sum_{i=1}^{n}(\alpha_i + \beta_i) = (n-2)\pi$.

The distance between v_i and m_{i-1} is $\epsilon_i / \cos\alpha_i$. Similarly, the distance between v_i and m_i is $\epsilon_i / \cos\beta_i$. Thus, the perimeter of P is equal to

$$per(P) = \sum_{i=1}^{n} \frac{\epsilon_i}{\cos\alpha_i} + \frac{\epsilon_i}{\cos\beta_i} \geq \epsilon \sum_{i=1}^{n} \frac{1}{\cos\alpha_i} + \frac{1}{\cos\beta_i}$$

By the strict convexity of $1/\cos\theta$ for $-\pi/2 < \theta < \pi/2$, this sum is uniquely minimized when all angles α_i and β_i are equal. To see this explicitly,

$$per(P) \geq 2n\epsilon \sum_{i=1}^{n} \frac{1/2n}{\cos\alpha_i} + \frac{1/2n}{\cos\beta_i} = 2n\epsilon E[1/\cos X]$$

where X is a random variable which takes value α_i with probability $1/2n$ and value β_i with probability $1/2n$ for $i \in \{1,\ldots,n\}$. By Jensen's inequality [7], $E[1/\cos X] \geq 1/\cos E[X]$ with equality if and only if X is constant. Since,

$$E[X] = \sum_{i=1}^{n} \frac{\alpha_i + \beta_i}{2n} = \frac{(n-2)\pi}{2n}$$

we have

$$per(P) \geq \frac{2n\epsilon}{\cos((n-2)\pi/2n)}$$

That is, the perimeter of P is at least the perimeter of the regular n-gon with tolerance ϵ, with equality if and only if P is the regular n-gon with tolerance ϵ. □

COROLLARY 1 *For $n > 3$, the convex n-gon with maximum tolerance within a given circle C is unique and is the regular n-gon with circumscribed circle C.*

PROOF. Let R_C be the regular n-gon with circumscribed circle C. Let P be any convex n-gon within C other than R_C. By theorem 1, there exists a regular n-gon R with the same tolerance as P and perimeter $per(R) \leq per(P)$. Since R_C has greater perimeter than any other n-gon within C [8], $per(P) < per(R_C)$. Since R_C is similar to R and has strictly greater perimeter, R_C has strictly greater tolerance than P. □

3 Open Problems

There is a nearly endless supply of interesting open problems of the form "What is the object within $\mathcal{O} \cap K$ whose $\mathcal{P}$-tolerance is maximum?" Two such problems posed in [2] are to find the set of n points within the unit disc whose Delaunay triangulation is most tolerant, and to find the n-gon within the unit disc which is "most simple," i.e. farthest from crossing itself. Alternatively, one could follow the approach presented in this paper and ask for the "smallest" object in $\mathcal{O}$ with $\mathcal{P}$-tolerance equal to ϵ.

Other interesting questions involve designing algorithms to calculate the $\mathcal{P}$-tolerance of a given object. (See [2] for a glimpse of what is known.)

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