Simple, Efficient and Robust Strategies to Traverse Streets

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Abstract

We present a family of strategies for the problem of searching in an unknown street for a target of unknown location. We show that a robot using a strategy from this family follows a path that is at most \( \pi + 1 \) times longer than the shortest possible path. Although this ratio is worse than the ratio of the best previously known strategy, which achieves a detour of at most \( 2\sqrt{2} \approx 2.8284 \) times the length of a shortest path, the simplicity of the analysis is interesting in its own. We use this new strategy as part of a hybrid method to obtain an equally simple strategy of slightly more complex analysis with a competitive ratio of \( \frac{1}{2}\sqrt{\pi^2 + 4\pi + 8} \approx 2.75844 \). The \( \pi + 1 \)-competitive strategy is very similar in spirit to the first published strategy of which the best analysis is very involved and gives a bound of \( \approx 4.44 \). More importantly, we show that the \( \pi + 1 \) strategy is robust under small navigational errors.

1 Introduction

One of the main problems in robotics is to find a path from the current location of the robot to a given goal of unknown location, particularly in those cases where the robot has only a partial knowledge of its surroundings.

In this paper we assume that the robot is equipped with a vision system that provides a visibility map of its local environment. Based on this information the robot has to find a path to a visually identifiable given goal that is located somewhere within the scene. The search of the robot can be viewed as an on-line problem in which the amount of information available to the robot increases as it discovers its surroundings in its travels. A natural measure of the quality of a search strategy is to use the framework of competitive analysis as introduced by Sleator and Tarjan [12]. A search strategy is called \( c \)-competitive if the path traveled by the robot to find the goal is at most \( c \) times longer than a shortest path. The parameter \( c \) is called the competitive ratio of the strategy.

Since there is no strategy with a competitive ratio of \( o(n) \) for scenes with arbitrary obstacles having a total of \( n \) vertices [2], the on-line search problem has been studied previously in various contexts where the geometry of the obstacles is restricted [1, 2, 3, 4, 10, 11].

Klein introduced the notion of a street which allowed for the first time a search strategy with a constant competitive ratio [7]. In a street, the starting points \( s \) and the goal \( g \) are located on the boundary of the polygon and the two polygonal chains from \( s \) to \( g \) are mutually weakly visible. Klein presents a strategy for searching in streets and gives an upper bound on its competitive ratio of \( 1 + 3/2\pi \approx 5.71 \). The analysis was recently improved to \( \pi/2 + \sqrt{1 + \pi^2/4} \approx 4.44 \) by Icking [6]. Though Klein’s strategy performs well in practice—he reports that no example had been found for which his strategy performs worse than 1.8—the strategy and its analysis are both quite involved and no better competitive ratio could be shown until, recently, Kleinberg presented a new approach.

In this paper we present a \( \pi + 1 \) \( \approx 4.14 \) analysis of a strategy similar to Klein’s. This analysis is significantly simpler than other published work [7, 9]. It also has the advantage that the strategy proposed is robust under small navigational errors. The simplicity of the strategy and analysis points naturally to possible improvements in the strategy. To illustrate this we present a hybrid method which
uses the $\pi + 1$-competitive strategy and results in a $\frac{1}{2}\sqrt{\pi^2 + 4\pi + 8} \approx 2.75844$ that better the best previously known of $2\sqrt{2}$-competitive ratio [8].

The rest of this paper is organized as follows. In the next section we introduce some notation and definitions. Then the family of strategies is described in Section 3, and in Section 4 we present its analysis. In the next to last section we present and analyze the hybrid strategy and then we conclude with some observations and directions of further research.

2 Definitions and Assumptions

We consider a simple polygon $P$ in the plane with $n$ vertices and a robot inside $P$ which is located at a start point $s$ on the boundary of $P$. The robot has to find a path from $s$ to the goal $g$. The search of the robot is aided by simple vision (i.e. we assume that the robot knows the visibility polygon of its current location). Furthermore, the robot retains all the information seen so far (in memory) and knows its starting and current position. We are, in particular, concerned with a special class of polygons called streets first introduced by Klein [7].

Definition 2.1 [7] Let $P$ be a simple polygon with two distinguished vertices, $s$ and $g$, and let $L$ and $R$ denote the clockwise and counterclockwise, resp., oriented boundary chains leading from $s$ to $g$. If $L$ and $R$ are mutually weakly visible, i.e. if each point of $L$ sees at least one point of $R$ and vice versa, then $(P, s, g)$ is called a street.

We denote the $L_2$-distance between two points $p_1$ and $p_2$ by $d(p_1, p_2)$ and the $L_2$-norm of a point $p$ by $\|p\|$.

Definition 2.2 Let $P$ be a street with start point $s$ and goal $g$. If $p$ is a point of $P$, then the visibility polygon of $p$ is the set of all points in $P$ that are seen by $p$. It is denoted by $V(p)$.

Definition 2.3 A window of $V(p)$ is an edge of $V(p)$ that does not belong to the boundary of $P$ (see Figure 1).

A window $w$ splits $P$ into a number of subpolygons $P_1, \ldots, P_k$ one of which contains $V(p)$. We denote the union of the subpolygons that do not contain $V(p)$ by $P_w$.

All windows are collinear with $p$. The end point of a window $w$ that is closer to $p$ is called the entrance point of $w$. We assume that a window $w$ has the orientation of the ray from $p$ to the entrance point of $w$. We say a window $w$ is a left window if the part $P_w$ of $P$ that does not contain $V(p)$ is locally to the left of $w$ w.r.t. the given orientation of $w$. A right window is defined similarly.

Definition 2.4 Two windows $w_1$ and $w_2$ are clockwise consecutive if the clockwise oriented polygonal chain of $V(p)$ between $w_1$ and $w_2$ does not contain a window different from $w_1$ and $w_2$. Counterclockwise consecutive is defined analogously.

3 A Family of Strategies

As observed by Kleinberg the shortest path $P$ from $s$ to $g$ consists of a number of line segments that touch reflex vertices of $P$. The general strategy we follow is to start at a reflex vertex $v$ of $P$ that belongs to $P$ and to identify another reflex vertex $v'$ of $P$ that is closer to $g$ by traveling further on. If the robot has identified $v'$, then it moves to it and starts the search anew. A move from one reflex vertex of $P$ on $P$ to another closer to $g$ is called a step.

If the robot has traveled along the path $P$, then we assume that the robot knows the part of $P$ that can be seen from $P$, i.e. the robot maintains the polygon $V(P) = \bigcup_{v \in P} V(p)$. We say a window $w$ of $V(p)$ is a true window w.r.t. $P$ if it is also a window of $V(P)$.

In the following we present the relevant results about true windows from [9].
In the case of two true windows, the robot has to determine the trajectory to follow. As proposed by Klein, the robot selects a target point $t_i$ in the line $p_i^+p_i^-$. The robot moves then in a straight line towards $t_i$ until either of the entrance points has changed or the goal has been identified and reached. There are several possible criteria to select $t_i$. Klein studied the case where $t_i$ balances the current absolute detour, as compared to the possible optimal trajectory, which is either the line joining the chain of left points $p_i^-$ or joining the right points $p_i^+$, depending on the actual location of the goal. Different criteria may be used to select the target point $t_i$. In particular, this leads to the main family of strategies:

**Strategy Walk-in-Circles.** Let $\ell$ be the subsegment of $p_i^-p_i^+$ which consists of the points $t$ such that $d(p_i^-t) \leq d(p_i^-, p_i)$ and $d(p_i^+t) \leq d(p_i^+, p_i)$ (see Figure 2). The algorithm chooses a point $t_i$ in the target segment $\ell$ and moves in a straight line towards it. If a new window appears, the robot recomputes $\ell$ according to the updated points $p_{i+1}^+$ and $p_{i+1}^-$, and the new position $p_{i+1}$, until the goal is found. (see Figure 3).

### 4 Analysis

We consider the case where the goal turns out to be on the right side. This is without loss of generality since the local target selection strategy is invariant under reflections.

The length of the trajectory traversed by the robot is determined by the sum of the length of all segments $p_ip_{i+1}$, i.e., $\sum_{i=0}^{n-1} d(p_i, p_{i+1})$, where $n$ is the number of extreme entrance points seen by the robot in a step. Note that $p_0 = s$ and $p_n^+ = g$. The length of each of these segments can be bounded by using the triangle inequality; viz. with notation as in Figure 3, we have that $d(p_i, p_{i+1}) \leq d(p_i, q) + d(q, p_{i+1})$, where $q$ is the point determined by the intersection of the line $p_{i+1}p_i^+$ and the circle centered at $p_i^+$ passing through $p_i$.

In turn the length of $d(p_i, q)$ is bounded by the length of the circular arc $pq_i$. Let $\alpha_i = \angle qpi^+p_i$ measured in radians. The length of the circular arc $pq_i$ is given by $\alpha_i \cdot d(p_i, p_i^+)$. Thus,

$$
\sum_i d(p_i, p_{i+1}) \leq \sum_i (d(p_i, q) + d(q, p_{i+1})) \\
\leq \sum_i (\alpha_i \cdot d(p_i, p_i^+) + d(q, p_{i+1})),
$$

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and the competitive ratio is determined by
\[
\frac{\sum_i d(p_i, p_{i+1})}{\text{Opt}} \leq \frac{\sum_i \alpha_i \cdot d(p_i, p_i^+) + \sum_i d(q, p_{i+1})}{\text{Opt}};
\]
where \(\text{Opt}\) is the length of the optimal walk from the starting point to the goal. Note that, if the final target is on the right chain as assumed, then \(\text{Opt} = d(p_0, p_0^+) + \sum_{i=0}^{n-1} d(p_i^+, p_{i+1})\).

The following two lemmas allows us to simplify the expression above. These lemmas follow quite naturally from the diagrams, and we provide a formal proof only for completeness.

**Lemma 4.1** Let \(D_i\) denote the length of the optimal walk from point \(p_i\) to the final target \(p_i^+\). Then \(D_{i+1} = D_i - d(q, p_{i+1})\).

**Proof:** If the target is located on the right side, the optimum trajectory from any given point is to move on a straight line to the uppermost visible point on the right chain, and follow the chain of points \(p_i^+ p_i^+\) from then onwards. From point \(p_i\), the length of the optimum trajectory is then \(D_i = d(p_i, p_i^+) + \sum_{j=i}^{n-1} d(p_j^+, p_{j+1}^+)\), and after moving to \(p_{i+1}\) is \(D_{i+1} = d(p_{i+1}, p_i^+) + \sum_{j=i}^{n-1} d(p_j^+, p_{j+1}^+) = d(p_i, p_i^+) - d(q, p_{i+1}) + \sum_{j=i}^{n-1} d(p_j^+, p_{j+1}^+) = D_i - d(q, p_{i+1})\) as required, since \(q\) is located on the circle centered at \(p^+\) and passing through \(p_i\).

At the starting position the distance \(D_0\) is precisely the length of the right chain walk \(d(p_0, p_0^+) + \sum_{i=0}^{n-1} d(p_i^+, p_{i+1}^+)\), and at the end of the walk the robot finds itself at a zero distance from the target point (i.e. the robot is at the target point).

Then the sum of the actual gains overall must be \(\sum_i d(q, p_{i+1}) = \sum_i d(p_i^+, p_{i+1}^+)\). Thus
\[
\frac{\sum_i d(p_i, p_{i+1})}{\text{Opt}} \leq \frac{\sum_i \alpha_i \cdot d(p_i, p_i^+) + \sum_i d(q, p_{i+1})}{\text{Opt}} = \frac{\sum_i \alpha_i \cdot d(p_i, p_i^+)}{\text{Opt}} + 1.
\]

The term \(\sum_i \alpha_i \cdot d(p_i, p_i^+)\) can be seen as a weighted sum, where the \(\alpha_i\)'s are the weights. Let \(\beta = \sum_i \alpha_i\), and let \(k\) be such that \(p_k\) is the point in the robot's trajectory such that \(d(p_k, p_k^+) \geq d(p_i, p_i^+)\). In other words, \(p_k\) denotes the largest term in the unweighted sum. Then we have \(\sum_i \alpha_i \cdot d(p_i, p_i^+) \leq \sum_i \alpha_i \cdot d(p_k, p_k^+) = \beta \cdot d(p_k, p_k^+)\).

**Lemma 4.2** The length of the polygonal chain \(\text{Opt} = d(p_0, p_0^+) + \sum_{i=0}^{n-1} d(p_i^+, p_{i+1})\) is no larger than the length of the polygonal chain \(\text{Opt} = d(p_0, p_0^+) + \sum_{i=0}^{n-1} d(p_i^+, p_{i+1})\).

We actually prove a stronger result, namely that \(d(p_i, p_i^+) \leq d(p_0, p_0^+) + \sum_{j=0}^{i-1} d(p_j^+, p_{j+1}^+)\).

**Proof:** By induction on the number of steps \(i\). When \(i = 0\), the two terms are equal and thus the inequality holds. For \(i+1\), we have that \(d(p_{i+1}, p_{i+1}^+) = d(p_i^+, p_{i+1}^+) + d(p_i^+, p_{i+1}) - d(p_{i+1}, q)\) (see Figure 3); and by induction hypothesis, \(d(p_{i+1}, p_{i+1}^+) \leq d(p_i^+, p_{i+1}^+) + d(p_0, p_0^+) + \sum_{j=0}^{i-1} d(p_j^+, p_{j+1}) - d(p_{i+1}, q) \leq d(p_0, p_0^+) + \sum_{j=0}^{i-1} d(p_j^+, p_{j+1})\). This implies
\[
\frac{\sum_i d(p_i, p_{i+1})}{\text{Opt}} \leq \frac{\beta \cdot d(p_k, p_k^+)}{\text{Opt}} + 1 \leq \beta + 1.
\]

Lastly, as it was noted by Klein (see proof of lemma 2.7 in [7]), if the angle \(\angle p_i^- p_i p_i^+\) ever exceeds \(\pi\) then at the point where the angle was \(\pi\) -- or possibly even before -- there must have been no true left window. In this case, the robot moves to the current \(p^+\) with competitive ratio bounded by \(\pi + 1\).

As a consequence, the trajectory can be analyzed in two parts. First, until the robot moves to the point \(p^+\) as a "temporal target", and second, the search afterwards, in which we start anew from a point on the right chain onwards towards the goal. The robot then recourses in the second search, and the total competitive ratio is bounded by the maximum of the competitive ratio on both parts.

Notice that \(\angle p_{i+1}^- p_i p_{i+1}^+ \geq \angle p_i^- p_i p_i^+ + \alpha_i\), and thus \(\pi \geq \angle p_i^- p_i p_i^+ \geq \angle p_0^- p_0 p_0^+ + \sum_i \alpha_i \geq \beta\). From
which follows that the competitive ratio for each part of the algorithm is, at worst,
\[
\sum_i \frac{d(p, p_{i+1})}{Opt} \leq \pi + 1.
\]

Theorem 4.3 A robot moving traveling under the strategy Walk-in-Circles has a \(\pi + 1\) competitive ratio.

As the target in each step is selected from the interval \(l\) this provides a margin of navigational error for the robot. That is, the strategy is robust under small constant bias of compass heading. The tolerance of the strategy is proportional to the aspect ratio of the smallest vs largest edges encountered and the smallest distinguished angle between left or right extreme entrance points.

5 A Hybrid Method

From the analysis above is clear that the competitive ratio of strategy Walk-in-Circles is directly dependent on the total "turn" angle \(\beta\). As it was pointed out, \(\beta\) is smaller than \(\pi\) minus the initial angle \(\angle p_0^+ p_0 p_0^-\). This implies that, if the initial angle is large, the strategy gives a better competitive ratio.

In this section we consider a hybrid method, in which a strategy similar to that proposed by Kleinberg [8] is followed for initial angles \(\angle p_0^- p_0 p_0^+\) smaller than \(\pi/2\) and the strategy of Section 3 is used for angles larger than \(\pi/2\).

Hybrid Strategy.

Cases 1-3 are as in Section 3.

Case 4 If \(\angle p_0^- p_0 p_0^+ \leq \pi/2\) then the robot moves on the line perpendicular to \(p_0^- p_0^+\). As the robot advances it updates the vertices \(p_i^-\) and \(p_i^+\) as the windows seen change. When either of \(\angle p_i^- p_i p_0\) or \(\angle p_i^+ p_i p_0 = \pi/2\), where \(p\) is the current position of the robot, it switches to strategy Walk-in-Circles, with \(p\) as starting point.

Case 5 If \(\angle p_0^- p_0 p_0^+ \geq \pi/2\) then the robot uses strategy Walk-in-Circles.

From the discussion in Section 4, it follows that cases 1-3 and 5 have a competitive ratio of at most \(\pi/2 + 1\). Case 4 requires a more careful analysis.

If, as in the previous section, we assume that the goal lies on the right side, then the optimal trajectory is given by \(d(p_0, p_0^+) + \sum_i d(p_i^+, p_{i+1}^-)\). Let \(j\) be the index of the reflex vertex in which the robot switched strategies. Notice that \(\angle p_j^+ p_0 p_j^-\) is now bigger equal to \(\pi/2\).

Lemma 5.1 The distance traversed by the robot up to the point where it switches strategy is bounded by \(d(p_0, p) \leq \sqrt{d(p_0, p_j^+)^2 - (p, p_j^+)^2}\) on either side.

Proof: For the vertex forming the right angle, the lemma follows trivially from the Theorem of Pythagoras. On the opposing vertex, say as in Figure 4, the law of the cosines states \(d(p_0, p_j^+)^2 = d(p_0, p)^2 + d(p, p_j^+)^2 - 2 d(p_0, p) d(p, p_j^+) \cos(\angle p_0 p p_j^+)\); which implies \(d(p_0, p_j^+)^2 \geq d(p_0, p)^2 + d(p, p_j^+)^2\) as \(\angle p_0 p p_j^+ \geq \pi/2\), from which the lemma follows.

As the robot applies strategy Walk-in-Circles as if \(p\) was the starting point, we have that the length of the distance traversed by it from \(p\) onwards is bounded by \((\pi/2 + 1) (d(p, p_j^+) + \sum_{i=j} d(p_i^+, p_{i+1}^-))\).

Thus the competitive ratio is given by \(R/Opt\) where,
\[
R = \sqrt{d(p_0, p_j^+)^2 - d(p, p_j^+)^2} + (\pi/2 + 1) \left( d(p, p_j^+) + \sum_{i=j} d(p_i^+, p_{i+1}^-) \right)
\]
\[
Opt = d(p_0, p_0^+) + \sum_{i=0}^{n-1} d(p_i^+, p_{i+1}^-).
\]

Let \(Opt' = d(p_0, p_j^+) + \sum_{i=j} d(p_i^+, p_{i+1}^-)\). Since \(Opt \geq Opt'\) then \(R/Opt \leq R/Opt'\). Without loss of generality, we can assume that \(d(p_0, p_j^+) = 1\). If \(R/Opt' \leq (\pi/2 + 1 + k)\) for some \(k \geq 0\), then \(R \leq (\pi/2 + 1 + k)Opt'\), which implies
\[
\sqrt{1 - d(p, p_j^+)^2} + (\pi/2 + 1) d(p, p_j^+) \leq (\pi/2 + 1 + k) \sum_{i=j} d(p_i^+, p_{i+1}^-).
\]
Since $k \cdot \sum_{i=1}^{n-1} d(p_i^+, p_{i+1}^+)$. can be arbitrarily small, for the expression above to be satisfied we need $\pi/2 + 1 - \sqrt{1 - d(p, p_j^+)^2} - (\pi/2 + 1) d(p, p_j^+) \geq -k$.

Let $f(x) = \pi/2 + 1 - \sqrt{1 - x^2} - (\pi/2 + 1) x$. This function has an absolute minimum in the domain of interest at $x_{\min} = (\pi + 2)/\sqrt{\pi^2 + 4\pi + 8}$ with $f(x_{\min}) = \pi/2 + 1 - \frac{1}{2} \sqrt{\pi^2 + 4\pi + 8}$. From which the fact that $k \geq \frac{1}{2} \sqrt{\pi^2 + 4\pi + 8} - \pi/2 - 1$ follows. Since the competitive ratio $R/Opt$ is bounded by $\pi/2 + 1 + k$, we have the following theorem.

**Theorem 5.2** A robot using the Hybrid Strategy has a $\frac{1}{2} \sqrt{\pi^2 + 4\pi + 8}$ competitive ratio.

The value $\frac{1}{2} \sqrt{\pi^2 + 4\pi + 8}$ is approximately 2.758...

6 Conclusions and Open Problems

We introduced and analyzed the first family of strategies for the street navigation problem. Because of this approach, the resulting algorithm is more robust under navigational error. It remains to be shown if it is possible for a robot to traverse a scene with a predetermined maximal navigational error per unit traversed at a predetermined competitive ratio. We also introduced a hybrid strategy which has a better competitive ratio than either of the original two strategies that define it. As the hybrid strategy shows, the $\pi + 1$ analysis not tight for small initial angles. In principle, it may be possible to improve on the $\pi + 1$ ratio by analyzing the Walk-in-Circles strategy differently for each case. As well, the best lower bound known for traversing streets is $\sqrt{2}$. The gap between this lower bound and the best upper bound is still significant, and remains to be improved.

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References


