Constructing Convex Hulls of Quadratic Surface Patches

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1 Introduction

We study geometric duality and use it to obtain an asymptotically efficient algorithm for constructing convex hulls of piecewise smooth objects in E^3. Specifically, we show that the convex hull of a collection of N smooth algebraic surface patches of bounded degree with bounded number of similarly constrained subfaces in E^3 has complexity $O(N^{2+\epsilon})$ for all $\epsilon > 0$ and can be constructed in randomized expected time of the same complexity. The dual construction also produces an $O(N \log N)$ algorithm for constructing the convex hull of an unordered collection of N algebraic curve segments and points in E^2.

For quadratic surface patches bordered by (an arbitrary number of) segments of conic sections, one can always obtain a closed form parameterizations of the the new surfaces which arise in the construction of the convex hull. Point classification against the hull may be performed in time $O(N \log N \log^* N)$ using a simple algorithm without explicit construction of the hull.

2 Object Model

For integer $1 \leq k \leq d$ an *open k-cell* is a subset of E^d homeomorphic with B^k. A *closed k-cell* is a subset homeomorphic with $\overline{B^k}$. We restrict our attention to open k-cells which are either diffeomorphic images of unit k-cubes (so that we can easily parameterize), or those which are also semi-algebraic sets. In either case a tangent space of dimension k is uniquely defined at each point in the open cell. Furthermore, the boundary of the cell can be partitioned into a finite collection of lower dimensional cells with similar properties. We call such cells ***smooth open k-cells*** and loosely use the term closed k-cells to refer to their closures. When the context makes the distinction between open and closed cells unimportant or obvious, we will also use the term *face* interchangeably with cell in accordance to the convention in computational geometry. [1] The faces that compose the boundary of face φ are called ***subfaces*** of φ.

Objects we deal with are finite *complexes* of such smooth faces. A complex K is a collection of open cells such that if $\varphi_1, \varphi_2 \in K$

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then $\overline{\varphi_1} \cap \overline{\varphi_2}$ are disjoint unions of open cells which are also in K (including the case for $\varphi_1 = \varphi_2$). Thus our objects need not be dimension-homogeneous as *subdivided manifolds* in [2] and need not be connected.

3 Duality

Recall that if K is a nonempty subset of E^d, then the *polar set* K^* of K is defined to be

$$K^* \equiv \{\mathbf{x} \in E^d \ : \ \mathbf{x} \cdot \mathbf{y} \leq 1 \text{ for all } \mathbf{y} \in K\} \ .$$

(See [3].) If the convex hull of K contains $\mathcal{O}$ in its interior, then $\mathbf{conv}K = K^{**}$. This is how we shall compute the convex hull of a complex.

This classical duality map has an intuitive interpretation. Given $K \subset E^d$, the set of hyperplanes in E^d can be partitioned with respect to K into

- $\mathcal{H}_c(K)$ the set of hyperplanes which cut K,
- $\mathcal{H}_s(K)$ the set of hyperplanes which support K. and
- $\mathcal{H}_b(K)$ the set of hyperplanes which bound but do not support K.

We also write $\mathcal{H}_{bs}(K)$ for $\mathcal{H}_b(K) \cup \mathcal{H}_s(K)$. Suppose the convex hull of K contains the origin $\mathcal{O}$ in its interior. If for each hyperplane in $\mathcal{H}_{bs}(K)$ we find the perpendicular foot dropped from the $\mathcal{O}$ and take the inversion¶, then we obtain the polar set K^*. Thus we shall represent a hyperplane H by by $\tilde{H}$, the inverted image of the perpendicular foot dropped from $\mathcal{O}$ to H.

For a smooth cell, the tangent hyperplanes afford an easier formulation from a combinatorial point of view. Let φ be a smooth k-face in E^d. Define $\mathcal{H}_{tan}(\varphi)$ to be the set of hyperplanes which contain the (k-dimensional) tangent space of some point $p \in \varphi$. Similarly $\mathcal{H}_{tan}(K)$ for a complex K is defined to be the union of $\mathcal{H}_{tan}(\varphi)$ for all faces φ of K. For a closed line segment in E^d, for example, $\mathcal{H}_{tan}$ and $\mathcal{H}_s$ coincide. In the more general case $\mathcal{H}_{tan}$ is a good "approximation" of $\mathcal{H}_s$. In fact,

$$\mathcal{H}_s(\overline{\varphi}) \subseteq \mathcal{H}_{tan}(\overline{\varphi}) \subseteq \mathcal{H}_s(\overline{\varphi}) \cup \mathcal{H}_c(\overline{\varphi})$$

¶*Inversion* in E^d is a point-to-point transformation which maps a vector $\mathbf{v}$ to the vector $\mathbf{inv v} = \mathbf{v}/|\mathbf{v}|^2$. [1]

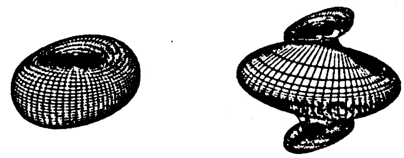

Figure 1: A dimpled surface and its dual. The two "umbrellas" turned inside out correspond to tangent planes to points in the two dimples. The apparent sharp cuspidal edges of the umbrellas are duals of two parabolic curves inside the dimples.

Our algorithm is based on the following

Theorem 3.1 *Let $K \subset E^d$ be a complex whose convex hull contains $\mathcal{O}$ in the interior. Then K^* is the (unique) convex cell in the arrangement of $\tilde{K}$ that contains $\mathcal{O}$ in its interior.*

The well-known fact that the new surfaces arising from convex hull construction in E^3 are developable becomes an immediate corollary since the intersection curves in the dual space dualize to developable surfaces.

Figure 1 shows a 3-d example — a "dimpled" object whose boundary is the zero set of

$$(4x^2 + 3y^2)^2 - 4x^2 - 5y^2 + 4z^2 - 1$$

(This equation is taken from [4].) Its dual is presented on the right. The two "umbrellas" turned inside out correspond to tangent planes to points in the two dimples, which do not contribute to the convex hull. The apparent sharp cuspidal edges of the umbrellas are duals to two parabolic curves inside the dimples. The convex hull is not completed by just a single common tangent plane on each side. Correspondingly, the self-intersections of this surface in the dual space consist of a pair of curves, dual to two developables that cover these dimples on the convex hull of the primal surface.

4 An Asymptotically Efficient Algorithm in E^3

The convex hull of a complex can thus be constructed in three major steps:

1. *Find dual for each face to form an arrangement of hypersurfaces in the dual space.* Each face can be processed independent of other faces except for its subfaces. Duals of faces need be broken at their singularities even though the primal faces are assumed to be smooth. This step takes time linear in the size of the input times an factor determined by the maximum degree of the input surfaces and curves.

2. *Find the polar set by constructing the cell containing the origin.* The polar set can be constructed using a number of cell construction algorithms such as [5, 6]. An asymptotically more efficient alternative is to reduce our problem of constructing a convex cell to that of the lower envelope of some other arrangement. Let e be the inward-pointing, unit normal of the boundary of an arbitrarily chosen halfspace H with $\mathcal{O}$ on the boundary. Consider the following function defined on the interior of H:

$$\mathcal{K} : \mathbf{x} \longmapsto \alpha^{-1}\mathbf{x} + \alpha\mathbf{e}$$

   where $\alpha = \mathbf{x} \cdot \mathbf{e}$. Its effect is to take vectors along the same direction to vectors ending in the same line parallel to $\mathbf{e}$, with their order on the original direction preserved. The boundary of half of the cell we wanted to construct now becomes the lower envelope and can be computed in time $O(N^{2+\epsilon})$ for all fixed $\epsilon > 0$ using the algorithm in [7]. Note that the topology of the surfaces on each hemisphere is unchanged by this transformation and that this transformation is rational in the arguments.

3. *Dualize the polar set.* Once we have the boundary of the polar set, we need not go over the first two steps again to obtain the convex hull in the primal space. Each face on the boundary can be dualized back individually. New edges are added between the duals of adjacent faces and the number of features of the dual graph is at most 6 times that of the polar boundary.

Note that we don't require much topological information as input since the dual is computed piece by piece.

Theorem 4.1 *Let K be a complex with N algebraic faces of bounded degree each with a bounded number of subfaces. Then the convex hull of K has $O(N^{2+\epsilon})$ faces for any fixed $\epsilon > 0$. It can be computed in randomized expected time $O(N^{2+\epsilon})$ for any $\epsilon > 0$ where the constant depends on the maximum degree of the polynomials in the input.*

5 Quadratic Surface Patches

The algebra simplifies significantly when the input is a *quadratic complex*, one whose 2-faces are conicoid (including plane) patches and whose 1-faces are segments of conic sections (including lines). The equations for the

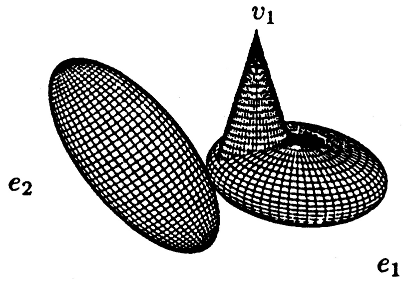

Figure 2: The union of two ellipsoids and a truncated cone.

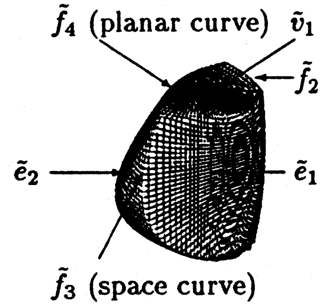

Figure 3: The polar set of the object, recovered as the cell containing the origin in the arrangement of dual surfaces. All dual faces are quadratic or planar.

duals are simpler, and it is easy to enumerate all possible singular cases.

Lemma 1 *Let σ be the quadratic hypersurface $\{\mathbf{x} \in P^d \,:\, \mathbf{x}^T A \mathbf{x} = 0\}$, where A is a real invertible matrix. Then*

$$\tilde{\sigma} = \{\tilde{\mathbf{x}} \,:\, \mathbf{x}^T B \mathbf{x} = 0\} \quad (1)$$

where B is A^{-1} with all elements in the first row and first column, except the element in the upper left corner, negated.

Lemma 2 *Every face on the convex hull of a quadratic complex has a closed form parameterization.*

Figure 2 shows two ellipsoids and part of a cone in space. The contribution of the cone to the convex hull is limited to its vertex. The

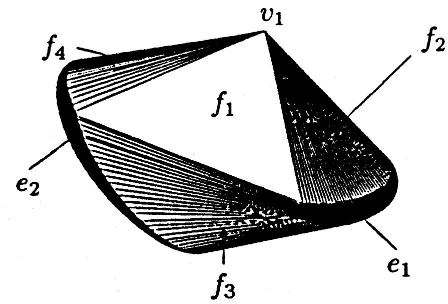

Figure 4: The convex hull of the original object, recovered from the dual of the polar dual (previous figure). Note that f_1 is planar since it is dual to the triple intersection (vertex) in the previous figure. Other new surfaces (f_2, f_3, and f_4) are developable, and arise from the corresponding edges of the polar set.

dual of the cone and that of its base circle are not shown, so as not to clutter the pictures. Curves of intersection have also been omitted, since they too do not contribute to the hull. The ellipsoids have implicit equations e_1 : $29x^2+36y^2+180z^2-24xy-48x+144y-36=0$ and e_2 : $225x^2+612y^2+388z^2-768yz-2448y+1536z+1548=0$. Their duals are $\tilde{e}_1$: $36x^2+9y^2+5z^2+24xy-20y-5=0$ and $\tilde{e}_2$: $100x^2-3y^2+153z^2+192yz+100y-25=0$. To find the polar set, we trim off the extraneous pieces of the surface patches, leaving alone the innermost cell in the dual space. It has three 2-faces as shown in Figure 3. Finally the polar set is dualized back to the primal space and the convex hull is obtained as in Figure 4.

6 Practical Considerations

Testing whether a point is within the convex hull of a quadratic complex can be performed using a straightforward algorithm in time $O(\lambda_4(N)\log N)$ without explicit construction of the hull. This can be useful if the sole purpose of constructing the convex hull is to classify a small number of points against the hull. Point classification translates to determining whether the intersection of a plane in the dual space with the polar set is empty (inside), singleton (on boundary), or otherwise (outside). This can be computed by a divide-and-conquer algorithm which constructs the intersection of halfspaces bounded by conic sections on the plane.

The convex hull can be rendered without explicit construction. In the dual space we choose a set of nicely spaced directions, for example, ones corresponding to the grid points on the unit sphere with chosen longitudes and latitudes. For each direction we shoot a ray. This ray is intersected with each dual surface to find the intersection point nearest the origin. The tangent plane to this dual surface is found, dualized, and then plotted. It is clear that if these directions form grid points on the boundary of the polar set such as the suggested choice, so will the plotted points in the primal space. Moreover, neighboring plotted points in the primal space have similar normal directions, a good choice for rendering. This technique is applicable to general complexes such as the dimpled surface in Figure 1, whose duals are too complicated to derive symbolically.

In case the hull needs to be constructed, existing simpler algorithms for converting CSG to boundary representations such as [8, 9] may be modified to work in the dual space. Although not as asymptotically efficient as the reduction to the lower envelop algorithm, these alternatives have been implemented or are easier to implement and have other uses in the primal space.

In CAD and CSG applications, being able to represent the original and derived surfaces and curves symbolically as opposed to numerically means a significant saving in storage. Better yet, one would usually like to have a parameterized representation available for rendering and other purposes. Fortunately we can do both quite efficiently with the duals and convex hulls of quadratic complexes. The fact that each conicoid is entirely either elliptic, parabolic, or hyperbolic also allows us to discard all parabolic and hyperbolic surface patches from the very beginning as their duals can never be part of the boundary of a convex cell by themselves.

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