Non-Stretchable Pseudo-Visibility Graphs

Ileana Streinu
Dept. of Computer Science,
Smith College,
Northampton, MA 01063, USA.
streinu@cs.smith.edu. *

Abstract

We exhibit a family of graphs which can be realized as pseudo-visibility graphs of pseudo-polygons, but not of straight-line polygons. The construction is based on the characterization of vertex-edge pseudo-visibility graphs of O'Rourke and Streinu[ORS96] and extends recent results on non-stretchable vertex-edge visibility graphs of Streinu [Str99]. We show that there is a pseudo-visibility graphs for which there exists only one of vertex-edge visibility graph compatible with it, which is then shown to be non-stretchable. The construction is then extended to an infinite family.

1 Introduction

Characterizing visibility graphs is a problem with a distinguished history (Ghosh[Gho88], Everett[Ev90], Abello and Kumar[AK95]), but so far several attempts to give good sets of conditions have been proved insufficient.

A different approach, introduced by O'Rourke and Streinu [ORS96] is to separate the combinatorial aspects of the problem from the questions of stretchability (known to be notoriously hard for pseudo-line arrangements, cf. Mnëv [Mn91] and Shor [Sh91]). They have introduced two new concepts: *vertex-edge visibility graphs* [ORS98] and *pseudo-visibility* [ORS96], and gave a complete combinatorial characterization of vertex-edge pseudo-visibility graphs. However, it is not clear a priori that the new class is any larger than just the class of straight-line vertex-edge visibility graphs, since it is conceivable that all such graphs can be realized with straight line edges (just as the class of planar graphs is realizable in this manner). In [Str99] a whole class of non-stretchable vertex-edge pseudo-visibility graphs is exhibited, thus settling this question. Moreover, it is shown that the stretchability question can in fact be decided efficiently for a class of vertex-edge pseudo-visibility graphs which includes these examples.

The original question was for visibility graphs, not vertex-edge visibility graphs. Since vertex-edge pseudo-visibility graphs contain more information than pseudo-visibility graphs, it is possible to have several vertex-edge pseudo-visibility graphs compatible with a given pseudo-visibility graph: some may be stretchable, some not. We are interested in the question: are there any pseudo-visibility graphs for which none of the compatible vertex-edge pseudo-visibility graph is stretchable? The visibility graphs of the non-stretchable pseudo-polygons from [Str99] are in fact compatible with straight line polygons, so the same family of examples does not work directly.

In this paper we exhibit a slightly more involved example of a pseudo-polygon with the property that its pseudo-visibility graph uniquely induces a vertex-edge visibility graph, which is then shown to be non-stretchable. This provides a strong separation between straight-line and pseudo visibility graphs. The example is then extended to an infinite family.

2 Preliminaries

Abbreviations: We may abbreviate the prefix *pseudo* by *p-* (as in *p-line* for pseudo-line), *vertex-edge pseudo-visibility graph* by *ve-graph*, *pseudo-visibility graph* by *v-graph* and *generalized configuration of points* by *gcp*. We use *ccw* for *counter-clockwise*.

An *arrangement of pseudolines* $\mathcal{L}$ is a collection of simple curves, each of which separates the plane, such that each pair of p-lines of $\mathcal{L}$ meet in exactly one point, where they cross.

Definition 2.1 *Let $V = \{v_0, v_1, \ldots, v_{n-1}\}$ be a set of points in the Euclidean plane $\mathbb{R}^2$, and let $\mathcal{L}$ be an arrangement of $\binom{n}{2}$ pseudolines such that every pair of points v_i and v_j lie on exactly one pseudoline $l_{ij} \in \mathcal{L}$,*

*Research partially supported by NSF RUI grant CCR-9731804 and a Smith College Picker Fellowship for Faculty Development.

and each pseudoline in $\mathcal{L}$ contains exactly two points of V. Then the pair $(V, \mathcal{L})$ is a generalized configuation of points *in general position.*

Two points a and b on a pseudoline $l \in \mathcal{L}$ determine a unique (closed) *segment ab* consisting of those points on l that lie between the two points. For $0 \le i \le n-1$, let $e_i = v_i v_{i+1}$ be the segment determined by v_i and v_{i+1} on $l_{i,i+1}$.[1]

Definition 2.2 *The segments $e_i = v_i v_{i+1}$ form a pseudo-polygon iff:*

1. *The intersection of each pair of segments adjacent in the cyclic ordering is the single point shared between them: $e_i \cap e_{i+1} = v_{i+1}$, for all $i = 0, 1, \ldots, n-1$.*
2. *Nonadjacent segments do not intersect: $e_i \cap e_j = \emptyset$, for all $j \ne i+1$.*

A p-polygon is a simple closed Jordan curve and separates the plane into two regions. We assume without loss of generality that the vertices of the p-polygon are numbered in ccw order, i.e. that the interior of the polygon lies to the left as the boundary is traversed in this order.

Pseudo-visibility is determined by the underlying arrangement $\mathcal{L}$: lines-of-sight are along pseudolines in $\mathcal{L}$.

Definition 2.3 *Vertex v_i* sees *vertex v_j ($v_i \leftrightarrow v_j$) iff either $v_i = v_j$, or they lie on a line $l_{ij} \in \mathcal{L}$ and the segment $v_i v_j$ is nowhere exterior to P.*

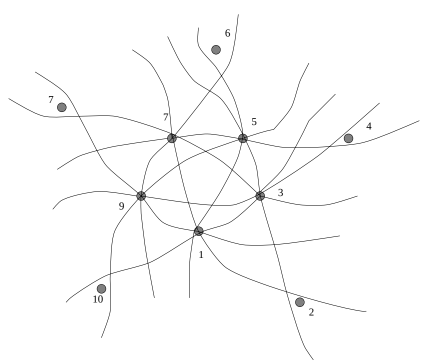

Figure 1: A non-stretchable generalized configuration of points.

Definition 2.4 *The* vertex-vertex pseudo-visibility graph *(v-graph) $G_V(P)$ of a p-polygon is a labeled graph with node set V, and an arc between two vertices iff they can see one another (according to Def. 2.3).*

[1] All index arithmetic is mod n throughout the paper.

We will often abbreviate $G_V(P)$ to G_V. Note that G_V is Hamiltonian: the arcs corresponding to the polygon boundary form a Hamiltonian circuit $(v_0, \ldots, v_{n-1})$. And also note that since G_V is labeled by V, which we assumed was labeled in a ccw boundary traversal order, the Hamiltonian circuit is provided by the labeling of the graph.

To define *vertex-edge pseudo-visibility* we need to define when a vertex sees an edge. This is based on the notion of a "witness" for a visible pair. Let $r_{ij} \subset l_{ij}$ be the ray directed from v_j not including v_i, closed at v_j.

Definition 2.5 *Vertex v_j is a* witness *for the vertex-edge pair (v_i, e) (and we say that v_i* sees *edge e) iff either*

1. *v_i is an endpoint of e, and v_j is also (here we permit $v_j = v_i$); or*
2. *v_i is not an endpoint of e, and*
   (a) *v_i sees v_j; and*
   (b) *the ray r_{ij} intersects e at a point p,*
   (c) *either $v_j = p$, or the segment $v_j p$ is nowhere exterior.*

We will refer to the line l_{ij} in the above definition as the *witness line.*

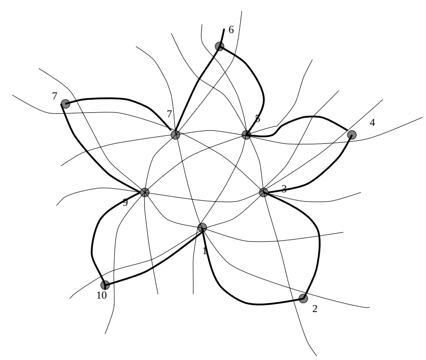

Figure 2: A non-stretchable pseudo-polygon.

Definition 2.6 *The* vertex-edge pseudo-visibility graph *G_{VE} of a polygon is a labeled bipartite graph with node node set $V \cup E$, and an arc between $v \in V$ and $e \in E$ iff v can see e (according to Def. 2.5).*

Notation: Let $P(i,j)$ be the open boundary interval containing all verices and edges of P encountered in a ccw traversal of the boundary of P from v_i to v_j. Similarly we define $P[i,j)$, $P(i,j]$ and $P[i,j]$ to include one or both endpoints of the interval.

The following lemma has been proved in [ORS96]:

Lemma 2.7 *If v_k sees non-adjacent edges e_i and e_j and no edge between, $v_k \in P[j+1, i]$, then exactly one of Case A or B holds:*

A *1. v_k sees v_{i+1} but not v_j.*

2. v_{i+1} is the right-witness for (v_k, e_j).

3. v_{i+1} sees e_j but v_j does not see e_i.

B *1. v_k sees v_j but not v_{i+1}.*

2. v_j is the left-witness for (v_k, e_i).

3. v_j sees e_i but v_{i+1} does not see e_j

One more concept needed is that of a "pocket":

Definition 2.8 *If v_i sees e_j and v_r and v_l are the right and left witnesses respectively, then $P[i, r)$ and $P(l, i]$ are the right and left* near pockets, *and $P(r, j]$ and $P[j+1, l)$ are the right and left* far pockets *of $v_i \to e_j$ respectively.*

The following lemma has been proved in [ORS96].

Lemma 2.9 *If v_i sees e_j and v_r and v_l are the right and left witnesses respectively, then*

1. *No vertex in the right near pocket sees an edge in the right far pocket.*
2. *No vertex in the right far pocket sees an edge in the right near pocket.*

Symmetric claims hold for the left pockets.

Lemma 2.10 *If v_i sees e_j and v_r and v_l are the right and left witnesses respectively, then v_r is an articulation point of the subgraph of G_{VE} induced by $P[i, j]$, and symmetrically v_l is an articulation point of the subgraph induced by $P[j+1, i]$.*

Theorem 2.11 *If G_{VE} is the vertex-edge visibility graph of a pseudo-polygon P, then it satisfies these two properties:*

1. *If v_k sees non-adjacent edges e_i and e_j and no edge between, $v_k \in P[j+1, i]$, then exactly one of these holds:*

   A. $(v_{i+1}, e_j) \in G_{VE}$, *or*

   B. $(v_j, e_i) \in G_{VE}$.

2. *In the two cases above, additionally:*

   A. *v_{i+1} is an articulation point of the subgraph of G_{VE} induced by $P[k, j]$.*

   B. *v_j is an articulation point of the subgraph of G_{VE} induced by $P[j+1, k]$.*

It has been shown in [ORS96] that these properties provide a complete characterization of vertex-edge pseudo-visibility graphs.

We now turn to stretchability questions. The basis of our construction comes from a classical example of a non-realizable allowable sequence, the so-called non-realizable pentagon (see [GP93]). The five central points in Figure 1 are pairwise connected by pseudo-lines. We force these pseudo-lines to cross such that we can then place the other exterior five points. We then draw extra pseudo-lines connecting all these pairs of points. This can be achieved in several ways, but each is a non-stretchable configuration of points.

To get an exampel of a non-stretchable pseudo-polygon, as in [Str99], we place a pseudo-polygon on top of this generalized configuration of points, as in Figure 2. No matter how the other pseudo-lines in the configuration meet, the basic internal structure remains the same: they all have the same vertex-edge visibility graph. Also, any pseudo-polygon with this ve-graph has to contain the non-realizable pentagon as un underlying subarrangement of its configuration of points.

3 An unstretchable vertex-vertex pseudo-visibility graph

Our goal is to extend this example to a non-realizable pseudo v-graph. We would like to get a v-graph with only one compatible ve-graph - the realizable one. But unfortunately the v-graph underlying the unstretchable ve-graph (Figure **??**) has both realizable and unrealizable compatible ve-graphs.

So we'll have to work harder to get our example. Notice that in general a v-graph can have many compatible ve-graphs (exponentially many, as shown in the full paper).

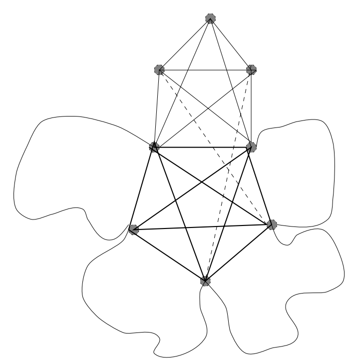

Figure 3: The gadget used in constructing the non-realizable v-graph.

The idea for obtaining a non-stretchable v-graph is to brake the symmetry, forcing the v-graph to have only one ve-graph. This ve-graph should lead to the non-stretchable pentagon. The construction is more complicated. It is based on the gadget in Fig. 3. The gadget is repeated at each of the five exterior vertices of the pentagon. The shick solid edges represent the common part of the v-graph, corresponding to the central mutually visible five vertices of the non-realizable pentagon. The thin solid and the thin dashed edges on the top are the actual gadget. The dashed edges are the symmetry brakers.

Let's prove that there is only one compatible ve-graph for this v-graph.

The argument is based on the properties of ve-graphs listed in the Preliminaries. Let's denote the relevant vertices of the gadget as in Figure 4: $v_1, v_2, b_2, w_2, t, w_1$ and b_1.

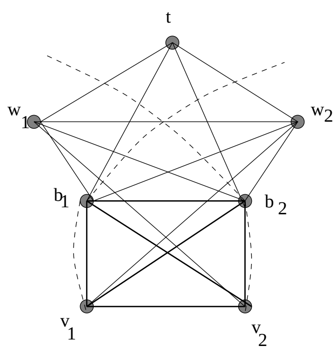

Figure 4: The labeled gadget.

Then since v_1 sees w_2, v_1 sees b_1 and no other vertex between these tow (in the ccw order induced by the boundary of the polygon), it follows that either b_1 or w_2 is an articulation point for t, when visibility from v_1 is considered. But w_2 can't be, since b_2 sees t (otherwise we would get a contradiction of the articulation point property of ve-graphs for b_2 in the near pocket, t in the far pocket, with articulation point w_2 and visibility from v_1.

So it follows that b_1 has to be articulation point for t with visibility from v_1. But then it follows that the pseudo-line through v_1b_1 extends to intersect the edge tw_2 (as in Figure 4).

A similar argument holds for the pseudo-line through v_2b_2. When we repeat this for all the five gadgets, we realize that we forced the unrealizable pentagon from Figure 1 as part of the underlying configuration of points for the ve-graph!

This completes the un-stretchable v-graph example. In the full paper we also show that this example can be extended to an infinite family.

4 Conclusion

We have shown that the class of pseudo v-graphs is strictly larger than the class of straight-line v-graphs. This result, together with the main characterization from [ORS96], yields a number of open problems for further research, which are described in the full paper.

References

[AK95] J. Abello and K. Kumar. Visibility graphs and oriented matroids. In R. Tamassia and I. G. Tollis, editors, *Graph Drawing (Proc. GD '94)*, volume 894 of *Lecture Notes in Computer Science*, pages 147–158. Springer-Verlag, 1995.

[Ev90] H. Everett. Visibility Graph Recognition. Ph.D. thesis, Computer Science Department, University of Toronto, 1989.

[Gho88] S. K. Ghosh. On recognizing and characterizing visibility graphs of simple polygons. In *Proc. 1st Scand. Workshop Algorithm Theory*, volume 318 of *Lecture Notes in Computer Science*, pages 96–104. Springer-Verlag, 1988.

[GP80] J.E. Goodman and R. Pollack. On the combinatorial classification of non-degenerate configurations in the plane. J. Combinat. Theory, ser. A 29 (1980), 220-235.

[GP93] J.E. Goodman and R. Pollack. Allowable sequences and Order Types in Discrete and Computational Geometry in *New Trends in Discrete and Computational Geometry*, J. Pach (ed.), Springer Verlag, 1993

[Mn91] N. Mnëv. The Universality Theorem on the Oriented Matroid Stratification of the Space of Real Matrices. in Discrete and Computational geometry, DIMACS Special Year (eds. J. Goodman, R. Polalck and W. Steiger), AMS/ACM DIMACS Series in Discrete Math. and Theoretical Computer Science, vol. 6, 1991.

[ORS96] J. O'Rourke, I. Streinu. Vertex-Edge Pseudo-Visibility Graphs: Characterization and Recognition, Proc. ACM Symp. Comput. Geometry, Nice (France), 1997.

[ORS98] J. O'Rourke, I. Streinu. The Vertex-Edge Visibility Graph of a Polygon, Comput. Geom. Theory Appl. 10 (105-120), 1998.

[Sh91] P. Shor. Stretchability of Pseudolines is NP-Hard. in *Applied Geometry and Discrete Mathematics, the Victor Klee Festscrift*, P. Gritzmann, B. Sturmfels (eds.), DIMACS Series in Discrete Math. and Theoretical Computer Science, vol. 4, 1991.

[Str99] I. Streinu. Stretchability of Star-like Pseudo-Polygons, Proc. ACM Symp. Comput. Geometry, Miami Beach, 1999 (to appear).