Small Weak Epsilon-Nets in Three Dimensions

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Abstract

We study the problem of finding small weak ε-nets in three dimensions and provide new upper and lower bounds on the value of ε for which a weak ε-net of a given small constant size exists. The range spaces under consideration are the set of all convex sets and the set of all halfspaces in $\mathbb{R}^3$.

1 Introduction

Let P be a set of n points in $\mathbb{R}^d$, and $\mathcal{R}$ be a family of subsets of $\mathbb{R}^d$, usually called *ranges*. A set $Q \subseteq P$ is called an *ε-net* for P (with respect to $\mathcal{R}$) if for every range $R \in \mathcal{R}$ with $|R \cap P| > \varepsilon n$ we have $R \cap Q \neq \emptyset$. If we allow Q to be any subset of $\mathbb{R}^d$ (instead of being just a subset of P), then Q is called a *weak ε-net* for P with respect to $\mathcal{R}$.

Hassler and Welzl [5] were the first who introduced the notions of ε-net and weak ε-net to computational geometry, and used these concepts to develop linear-size data structures for certain range query problems. Later on, these concepts found many applications in other problems in geometric optimization and approximation algorithms.

It is well-known that for the range spaces of a finite VC-dimension c, ε-nets of size $O(\frac{c}{\varepsilon}\log\frac{c}{\varepsilon})$ exist [5]. In fact, any random sample of P of this size is simply an ε-net for P with probability close to 1. For range spaces of infinite VC-dimension, the previous result no longer applies. However, when the range space is the set of all convex subsets of $\mathbb{R}^d$, Alon *et al.* [1] has shown that weak ε-nets of size $O(1/\varepsilon^{d+1-\delta_d})$ always exist, where δ_d is a positive number tends to zero as $d \to \infty$. This bound has later been improved to $O(1/\varepsilon^d \text{polylog}(1/\varepsilon))$ [3, 6].

Recently, Aronov *et al.* [2] have studied small weak ε-nets in two dimensions, and have provided various upper and lower bounds on ε for which weak ε-nets of small constant size exist. The main tools they have used in the construction of their upper bounds are ham-sandwich cuts and centerpoints. In this paper, we show that a generalized version of the ham-sandwich cut, which we call *unbalanced partitioning*, can be used to get improved upper bounds in two dimensions. We then use this tool to provide some new results on the small weak ε-nets with respect to convex sets in three dimensions. We also study small weak ε-nets with respect to the set of all halfspaces in $\mathbb{R}^3$.

2 Preliminaries

For any range space $\mathcal{R}$, we define $\varepsilon_i^{\mathcal{R}}$ to be the minimum real number, between 0 and 1, such that for any point set P in $\mathbb{R}^d$, there exists a weak ε-net of size i for P with respect to $\mathcal{R}$. In this paper, when $\mathcal{R}$ is the family of all convex sets in $\mathbb{R}^3$, we drop $\mathcal{R}$ from our notations and simply write ε_i instead of $\varepsilon_i^{\mathcal{R}}$.

The *centerpoint* of an n-point set $P \subset \mathbb{R}^d$ is a point c such that any convex set containing more than $\frac{d}{d+1}n$ points of P contains c. It is known that any point set in $\mathbb{R}^d$ admits a centerpoint [4].

Given d sets $P_1, \ldots, P_d$ in $\mathbb{R}^d$, a *ham-sandwich cut* is a hyperplane that simultaneously bisects all the P_i's. The *ham-sandwich theorem* [4] guarantees the existence of such a cut.

Using ham-sandwich cuts, one can partition any point set P in the plane into four subsets of equal size, or into two pair of subsets, such that each pair consists of subsets of equal size. In this paper, we use a more general form of the partitioning of the plane, which we call *unbalanced partitioning*, as stated in the following theorem.

Theorem 1 *Given an n-point set $P \subset \mathbb{R}^2$ and four positive numbers $\alpha_1, \ldots, \alpha_4$ such that $\sum_{i=1}^4 \alpha_i = 1$, there exist two lines ℓ and ℓ' that partition P into four subsets $A_1, \ldots, A_4$ such that for $1 \leq i \leq 4$, A_i contains at most $\alpha_i n$ points of P.*

Proof. Let ℓ be a vertical line that partitions P into two subsets A and B of sizes at most $(\alpha_1+\alpha_2)n$ and $(\alpha_3+\alpha_4)n$, respectively. We may assume that A and B are two positive density functions on the plane whose domains are bounded, connected, and separated by the line ℓ. We also assume that A is to the left of ℓ. For any point p on ℓ, we denote by $\ell_l(p)$ (resp. $\ell_r(p)$) the line that goes through p and divides A (resp. B) in ratio $\alpha_1 : \alpha_2$ (resp. $\alpha_3 : \alpha_4$). If we move p up ℓ from bottom to top, the slope of $\ell_l(p)$ continuously increases

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from $-\infty$ to $+\infty$, while the slope of $\ell_r(p)$ continuously decreases from $+\infty$ to $-\infty$. Thus, at some point p, two lines $\ell_l(p)$ and $\ell_r(p)$ coincide, which gives the second line ℓ' making the desired partition. □

It is known that for any n-point set $P \subset \mathbb{R}^3$, there exist three planes that divide P into eight subsets, each containing at most $n/8$ of the points of P [7]. We can slightly modify the proof of this eight-partition theorem to get the following restricted unbalanced partitioning in three dimensions.

Theorem 2 *Let A be a positive density function over a bounded connected region C in $\mathbb{R}^3$ such that the total mass of A over C is 1. For any positive number $\alpha \in [0, \frac{1}{2}]$, there exist three planes that partition A into eight regions, such that two adjacent regions contain equal mass α and the other six regions contain equal mass $(1-2\alpha)/6$.*

The proof of the above theorem is analogous to the proof of the eight-partition theorem (refer to [7] for the details of the proof). We just note that for $\alpha = 1/8$, Theorem 2 is equivalent to the original eight-partition theorem.

3 Convex Ranges

3.1 Warm-Up: An Improved Bound in 2D

To demonstrate the usefulness of the unbalanced partitioning, we start by giving a new upper bound for $\varepsilon_4^{\mathcal{C}}$, where $\mathcal{C}$ is the family of all convex sets in $\mathbb{R}^2$. Aronov *et al.* [2] have previously given an upper bound of $\frac{4}{7}$ on the value of $\varepsilon_4^{\mathcal{C}}$, using traditional ham-sandwich cuts. We show that our approach of using unbalanced partitioning can lead to a better bound as follows.

Theorem 3 *If $\mathcal{C}$ is the family of all convex sets in $\mathbb{R}^2$, then $\varepsilon_4^{\mathcal{C}} \le \frac{6}{11}$.*

Proof. Let P be any n-point set in the plane. According to Theorem 1, we can partition P by two lines ℓ and ℓ' into four subsets $A_1, \ldots, A_4$ such that A_1 contains at most $\frac{2n}{11}$ points, and the other three subsets A_2, A_3, and A_4, each contains at most $\frac{3n}{11}$ points of P. Let q_1 be the intersection point of ℓ and ℓ', and let q_i, for $2 \le i \le 4$, be the centerpoint of A_i. We show that $Q = \{q_1, \ldots, q_4\}$ is a weak $\frac{6}{11}$-net for P.

Let C be any convex set that avoids Q. Since C does not contain q_1, it must avoid at least one of the four subsets $A_1, \ldots, A_4$. Assume first that C avoids A_1. Since C does not contain $q_2, \ldots q_4$, by the property of the centerpoints, it contains at most $\frac{2}{3} \cdot \frac{3n}{11}$ points from each of $A_2, \ldots, A_4$. Thus, in this case, C contains at most $3 \cdot \frac{2}{3} \cdot \frac{3n}{11} = \frac{6}{11}n$ points of P.

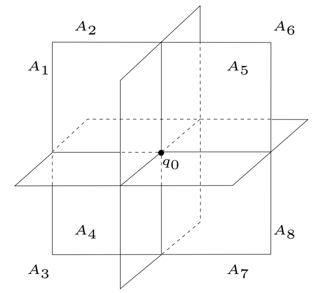

Figure 1: Partitioning a point set into eight subsets $A_1, \ldots, A_8$ by three planes.

Now, assume that C avoids some other subset, say A_2. It follows that C avoids all the $\frac{3n}{11}$ points of A_2 as well as $\frac{1}{3} \cdot \frac{3n}{11}$ points from each of A_3 and A_4. Thus, in this case, C avoids $(\frac{3}{11} + 2 \cdot \frac{1}{3} \cdot \frac{3}{11})n = \frac{5}{11}n$ points of P. Therefore, in any case, at most $\frac{6}{11}n$ points of P can lie in C. □

3.2 Upper Bounds in Three Dimensions

Now, we turn to the upper bounds in three dimensions. Here, the range space under consideration is the family of all convex sets in $\mathbb{R}^3$. By the centerpoint theorem, we have $\varepsilon_1 = \frac{3}{4}$. We show how to obtain upper bounds better than $\frac{3}{4}$, when we are allowed to choose more than one point. In this subsection, we use the following terminology. Let P be any n-point set in $\mathbb{R}^3$. We partition P by three planes into eight subsets $A_1, \ldots, A_8$, as shown in Figure 1. We denote the intersection point of the three planes by q_0.

Theorem 4 $\varepsilon_3 \le \frac{19}{26}$*,* $\varepsilon_4 \le \frac{5}{7}$*, and* $\varepsilon_5 \le \frac{11}{16}$*.*

Proof. We first prove that $\varepsilon_3 \le \frac{19}{26}$. Using Theorem 2, we partition P into eight subsets $A_1, \ldots, A_8$ such that each of A_1 and A_2 contains at most $\frac{n}{26}$ points, and each of the subsets $A_3, \ldots A_8$ contains at most $\frac{2n}{13}$ points of P. Let $q_1 = \text{centerpoint}(A_3 \cup A_5 \cup A_7)$ and $q_2 = \text{centerpoint}(A_4 \cup A_6 \cup A_8)$. We show that $Q = \{q_0, q_1, q_2\}$ is a weak $\frac{19}{26}$-net for P.

Consider a convex set C that avoids Q. Since C does not contain q_0, it must avoid at least one of the eight subsets $A_1, \ldots, A_8$. Assume that C avoids A_1. Since C does not contain q_1, by the property of the centerpoints it contains at most $\frac{3}{4} \cdot \frac{6n}{13}$ points from $A_3 \cup A_5 \cup A_7$. Similarly, C contains at most $\frac{9n}{26}$ points from $A_4 \cup A_6 \cup A_8$. Since C can contain all the $\frac{n}{26}$ points of A_2, the maximum total number of points that C contains is $(2 \cdot \frac{9}{26} + \frac{1}{26})n = \frac{19n}{26}$.

Now, assume that C avoids one of the larger subsets, say A_3. It follows that C avoids all the $\frac{2n}{13}$ points of A_3 as well as $\frac{1}{4} \cdot \frac{6n}{13}$ points from $A_4 \cup A_6 \cup A_8$. Thus, C totally avoids $(\frac{2}{13} + \frac{1}{4} \cdot \frac{6}{13})n = \frac{7n}{26}$ points in this case.

So, in any case, C can not contain more than $\frac{19n}{26}$ points of P, and hence, Q is a weak $\frac{19}{26}$-net for P.

Next, we prove that $\varepsilon_4 \leq \frac{5}{7}$. Here, we partition P such that A_1 and A_2 each contains $\frac{n}{14}$ points, and the other subsets $A_3, \dots A_8$ each contains $\frac{n}{7}$ points of P. Let $q_1 = \text{centerpoint}(A_3 \cup A_4)$, $q_2 = \text{centerpoint}(A_5 \cup A_6)$, and $q_3 = \text{centerpoint}(A_7 \cup A_8)$. We show that the set $Q = \{q_0, \dots, q_3\}$ is a weak $\frac{5}{7}$-net for P.

Any convex set C that avoids Q must avoid at least one of the eight subsets $A_1, \dots, A_8$. Two cases arise. The first case is when C avoids one of the smaller subsets, say A_1. Since C does not contain q_1, by the property of the centerpoints it contains at most $\frac{3}{4} \cdot \frac{2n}{7} = \frac{3n}{14}$ points from $A_3 \cup A_4$. Similarly, C contains at most $\frac{3n}{14}$ points from each of $A_5 \cup A_6$ and $A_7 \cup A_8$. Furthermore, C can contain all the points of A_2. Therefore, the total number of points that C contains is at most $(3 \cdot \frac{3}{14} + \frac{1}{14})n = \frac{5n}{7}$ points of P.

The second case is when C avoids one of the larger subsets, say A_3. Then C avoids all the $\frac{n}{7}$ points of A_3 as well as $2 \cdot \frac{1}{4} \cdot \frac{2n}{7}$ points from $A_5 \cup A_6$ and $A_7 \cup A_8$. Thus, C totally avoids $(\frac{1}{7} + \frac{1}{7})n = \frac{2n}{7}$ points in this case. Therefore, in both cases, C can not contain more than $\frac{5n}{7}$ points of P.

To prove the upper bound $\varepsilon_5 \leq \frac{11}{16}$, we partition the point set P into eight subsets of equal size, using the eight-partition theorem. For $1 \leq i \leq 4$, define $q_i = \text{centerpoint}(A_{2i-1} \cup A_{2i})$. Let $Q = \{q_0, \dots, q_4\}$. Any convex set C that avoids Q avoids at least one of the eight subsets, say A_1. By the property of the centerpoints, C contains at most $\frac{3}{4} \cdot \frac{n}{4} = \frac{3n}{16}$ points from each of $A_3 \cup A_4$, $A_5 \cup A_6$ and $A_7 \cup A_8$. Furthermore, C can contain all the points of A_2. Therefore, C contains at most $(3 \cdot \frac{3}{16} + \frac{1}{8})n = \frac{11n}{16}$ points of P, showing that Q is a weak $\frac{11}{16}$-net for P. □

We can recursively use the construction for ε_5 to obtain a weak ε-net of size $O(1/\varepsilon^5)$ for $\frac{1}{2} \leq \varepsilon \leq 1$. In general, this upper bound is weaker than the best known upper bound $O(\frac{1}{\varepsilon^3}\text{polylog}\frac{1}{\varepsilon})$ [3, 6]. However, for small values of ε, the actual size of the weak ε-net constructed by our method is smaller then the one constructed by the previous general methods. For example, with $i = 5$ points we can get a weak $\frac{11}{16}$-net, while the previous methods in [3, 6] need at least 9 points to obtain such a weak ε-net.

Remark. Applying the method used in Theorem 4, we can easily obtain two other upper bounds $\varepsilon_8 \leq \frac{2}{3}$ and $\varepsilon_9 \leq \frac{21}{32}$. For the bound $\varepsilon_8 \leq \frac{2}{3}$, we partition P such that the two subsets A_1 and A_2 each contains $\frac{n}{6}$ points and the other six subsets each contains $\frac{n}{9}$ points of P. The set consisting of the centerpoint of $A_1 \cup A_2$ as well as the centerpoints of the other six subsets plus the point q_0 forms a weak $\frac{2}{3}$-net for P. For the bound $\varepsilon_9 \leq \frac{21}{32}$, we use an eight-partition of P and then selects the centerpoint of each subset as well as q_0 to obtain the desired weak $\frac{21}{32}$-net.

4 Halfspace Ranges

In this section, we study $\varepsilon_i^{\mathcal{H}}$, where $\mathcal{H}$ is the family of all halfspaces in $\mathbb{R}^3$. It is clear from the centerpoint theorem that $\varepsilon_1^{\mathcal{H}} = \frac{3}{4}$.

Theorem 5 $\frac{3}{5} \leq \varepsilon_2^{\mathcal{H}} \leq \frac{2}{3}$.

Proof. We first prove that $\varepsilon_2^{\mathcal{H}} \leq \frac{2}{3}$. Consider an n-point set $P \subset \mathbb{R}^3$. Let W_1 and W_2 be two planes parallel to the xy-plane that bound P, i.e. all the points of P lie in between W_1 and W_2. Let P_1 and P_2 be the vertical projections of P on W_1 and W_2, respectively. Define $q_1 = \text{centerpoint}(P_1)$ and $q_2 = \text{centerpoint}(P_2)$. We argue that the set $Q = \{q_1, q_2\}$ is a weak $\frac{3}{2}$-net for P.

Let h be any halfspace that avoids Q. Define $h_1 = h \cap W_1$ and $h_2 = h \cap W_2$. Let n_1 be the number of points of P_1 contained in h_1. Since h_1 avoids q_1, we have by the property of the centerpoints that $n_1 \leq \frac{2n}{3}$. Similarly, if n_2 is the number of points of P_2 contained in h_2, we have $n_2 \leq \frac{2n}{3}$. But, it is easy to see that h contains at most $\max\{n_1, n_2\}$ points of P. Therefore, h contains at most $\frac{2n}{3}$ points, and hence Q is a weak $\frac{2}{3}$-net for P.

Now, we prove the lower bound $\varepsilon_2^{\mathcal{H}} \geq \frac{3}{5}$. To this aim, for any n, we construct a set P of n points such that for any pair of points p and q, there exists a halfplane that avoids both the points p and q and contains at least $\frac{3}{5}n$ points of P.

Figure 2 shows the construction of such a set P. Each point in this figure represents a ball of sufficiently small radius containing $\frac{n}{20}$ points. We denote this set of balls by B, and refer to its members simply as the points of B. The points of B are arranged in five groups, named a, b, c, d and e, each containing four points on the boundary or inside a tetrahedron T of unit side length. Each of the outer four groups, a, b, d and e, consists of a point on a

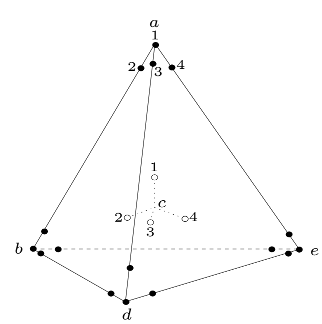

Figure 2: The point set used to prove the lower bound for $\varepsilon_2^{\mathcal{H}}$.

vertex v of T, as well as three points at distance δ from v on the edges connecting v to the other three vertices of T. The central group, c, consists of four points at distance γ from the center of T on the lines connecting the center to the four vertices of T. (In our example in Figure 2, we have chosen $\delta = 1/10$ and $\gamma = 1/5$).

We show that for any two given points p and q, there is a plane H going through p and q, such that at least 12 points out of the 20 points of B lie in an open half-space bounded by H. We may assume without loss of generality that p and q coincide with two points of B.

Assume first that both p and q are in outer groups of B, say in a and b. Consider the plane H that goes through p and q, and is perpendicular to the face abe of T. Then, it is easy to verify that the groups c, d and e are fully contained in one side of H, and hence, H has 12 points in one side.

Now, assume that one of the points, say p, is in an outer group, say in a, and the other point, q, is in the central group, c. Two cases arise:

Case (1) p is the topmost point of a, i.e. $p = a_1$ (see Figure 2). If $q \in \{c_1, c_2\}$, then we choose H as the plane that contains (p, q) and is parallel to the edge de of T. We can easily see that two groups d and e as well as the set $\{a_3, a_4, c_3, c_4\}$ are fully contained in one side of H. The case when $q \in \{c_3, c_4\}$ is analogous.

Case (2) p is a side point of a, say a_2. If $q \in \{c_1, c_2\}$, then the plane that contains (p, q) and is parallel to the edge de has the entire groups d and e as well as the set $\{a_1, a_3, a_4, c_3, c_4\}$ fully contained in one side. Now, assume that $q = c_3$. By choosing proper values for δ and γ (as what are chosen in our example), we are sure that the plane H going through three point p, q and c_4 contains the entire groups d and e as well as the set $\{a_1, a_3, a_4\}$. Now, we rotate H slightly around the line pq such that c_4 lies in the same side of H that the other two groups d and e lie. Figure 3 shows the resulting plane H that contains 12 points in one side. The case when $p = a_2$ and $q = c_4$ is analogous. □

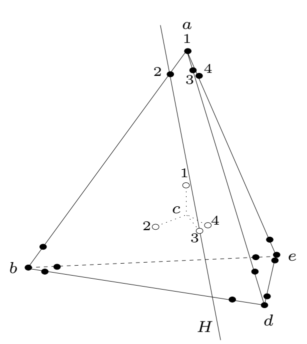

Figure 3: A plane H going through a_2 and c_3, containing 12 points in one side.

Theorem 6 $\varepsilon_3^{\mathcal{H}} = \frac{1}{2}$.

Proof. We first prove that $\varepsilon_3^{\mathcal{H}} \leq \frac{1}{2}$. Consider an n-point set $P \subset \mathbb{R}^3$. Let W be a plane that bisects P. We project all the points of P into W to obtain a two dimensional set P'. Let Q be the set of vertices of a triangle that bounds P'. It is easy to verify that any halfplane avoiding Q contains at most half the points of P, and therefore, Q is a weak $\frac{1}{2}$-net for P.

To prove the lower bound, consider an n-point set $P \subset \mathbb{R}^3$ in general position. For any given set of three points Q, one of the two open halfplanes whose bounding plane is incident to the points of Q contains at least half the points of P, and hence, $\varepsilon_3^{\mathcal{H}} \geq \frac{1}{2}$. □

5 Conclusions

In this paper, we have presented several new upper and lower bounds for $\varepsilon_i^{\mathcal{R}}$ for the small values of i, where $\mathcal{R}$ is the family of all convex sets, or the set of all halfplanes in $\mathbb{R}^3$. The natural open question is whether we can find alternative techniques for constructing weak ε-nets to improve the upper bounds provided in this paper. In particular, it is not possible to get any upper bound better than the trivial $\frac{3}{4}$ bound for ε_2, using the techniques currently used for constructing small weak ε-ntes. Reducing the gap between the upper and the lower bound for ε_2^H given in Section 4 is also an interesting question that remains open.

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