Two-Guard Art Gallery Problem

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Abstract

We define and study the two-guard art gallery problem, i.e. guarding a gallery so that each point is visible to at least two guards. We prove that this problem is NP-hard for vertex guards. We present a polynomial time approximation algorithm that produces solutions within $O(\log n)$ of optimum.

1 Introduction

More than three decades ago, Klee and Chvatal defined the following problem: *Given a polygon with n vertices, determine the smallest number of guards necessary to cover it.* This problem is known as the art gallery problem, and has become one of the central problems about visibility in computational geometry. This problem has a wide range of applications in the real world, such as building security, motion capture, robot path planning, to name a few. The two-guard art gallery problem is a new variant of the traditional art gallery problem mentioned above, and we define it as follows:

> *Given a polygon with n vertices, determine the smallest number of guards necessary to cover it such that any point in this polygon is seen by at least two guards.*

The two-guard art gallery problem is motivated by real applications in motion capture experiments. Under certain situations, people use two cameras to locate some moving object's position in the guarding area (surveillance region) as the intersection of rays from these cameras. A practical limitation of this approach is that the object's position can not be determined when the rays are collinear.

The outline of this paper is as follows. In section 2 we study the computational complexity of finding minimum vertex guard covers for the two-guard art gallery problem. In section 3 we present a polynomial time approximation algorithm that produces solutions within $O(\log n)$ of the optimum. In section 4 we discuss the co-linearity issue.

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1.1 Previous Work

After Chvatal's art gallery theorem was proposed in 1975 [3], researchers came up with many important results (please see O'Rouke's book[11] and Shermer's summary [14]). Several variants of the traditional art gallery problem have been studied. Researchers investigated different types of polygons (such as polygons with holes, orthogonal polygons [7], [4], [9], [12], [13], histograms [2], spiral polygons [10], etc), and also different types of guards (e.g., point guards, vertex guards, edge guards, moving guards, etc). The computational complexity of the classic art gallery problem was studied in [8] where it was shown to be NP-hard for vertex guards; the result was extended to point guards in [1]. Most variants of the classic problem were proven to be NP-hard. An approximation algorithm for finding a minimum vertex guard cover was given in [5].

1.2 Preliminaries

In this paper, we study simple polygons without holes. Guarding means covering the interior space. We present results for only vertex guards, which means restricting guards to be on vertices. Each vertex can have at most one guard on it.

2 Computational Complexity

Since finding an optimal set of vertex guards for the traditional art gallery problem is known to be NP-hard [8], it is natural to ask if this new variant is also NP-hard. The answer is yes. However, the trivial approach of proving NP-hardness by doubling vertices in the polygon of Lee and Lin's proof (split every possible vertex guard to two distinct vertices such that they are extremely close and have same visibility) does not solve the problem because it is hard to deal with concave vertices. Check the visibility of vertex c and c' in Figure 1 as an example.

Theorem 1 *Finding the minimum number of vertex guards such that any point of the polygon is covered by at least two guards is NP-hard.*

Proof. The proof is by polynomial time reduction from 3-SAT to the two-guard art gallery problem. Let a 3-SAT instance have k clauses and n variables. We modify the proof by Lee and Lin for the traditional art gallery

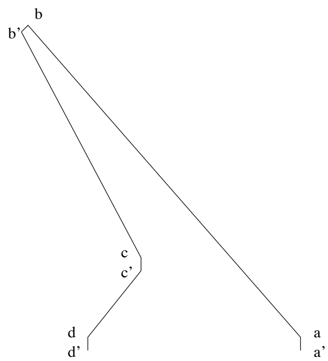

Figure 1: Doubling vertices of literal pattern in Lee and Lin's proof

problem with vertex guards [8]. In this paper, we refer to [8] for the complete construction of the proof, and show only the modifications and differences which have to be made.

We use the "fork-shaped" gadget shown in Figure 2 extensively in the proof. The function of this gadget is to force a particular vertex to be selected as the location of a guard. For example, in Figure 2, in order to guard the two rectangular spikes in the "fork", the bottom point (suppose it is a vertex) must be chosen in order to minimize the number of guards. The rectangular spikes can be tuned so that only one vertex in the overall structure can cover both spikes. Then, one additional vertex in the interior of each rectangular spike will need to be chosen to make this fork shape two-guard coverable. Therefore, for each fork-shaped structure, we need a minimum of 3 guards.

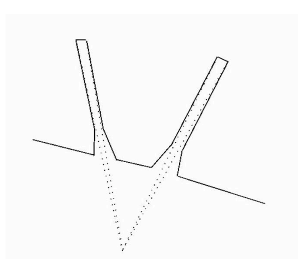

Figure 2: Small "fork-shaped" gadget

We design the literal pattern and one 3-CNF clause as in Figure 3(a) and (b) respectively. Given a fixed vertex guard e, each literal pattern structure needs one additional vertex to cover the ditch in the upper area. As in Lee and Lin's proof [8], there are only two feasible vertices (a and b) which can be chosen as guard locations. Selecting the inner vertex b corresponds to an assignment of *"False"* to the literal while selection of the outer vertex a corresponds to assigning *"True"* to the literal. These two special vertices have visibilities to some "well-shaped" area in the variable pattern structures. There are 10 distinguished points in each clause structure as marked in part (b) of Figure 3. A minimum vertex guard set uses 7 of them to cover each clause structure. The main body of the clause structur must be guarded by one of 2, 4, 6 as well as 8(check the lower right corner as an example).

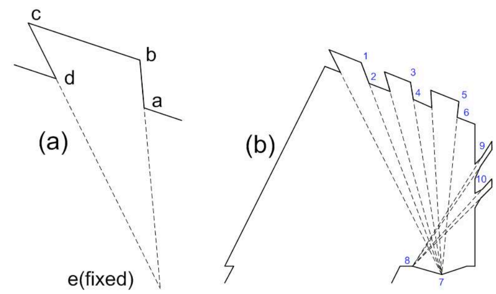

Figure 3: (a) literal pattern(b) one clause structure

The treatment of each variable and its negation is similar to that in Lee and Lin's proof.

If a variable occurs in l clauses, we add $2l + 1$ "fork-shaped" gadgets in its corresponding variable pattern structure as in Figure 4, in which l =1. Spike p and spike q are visible to two distinguished vertices in one literal pattern respectively.

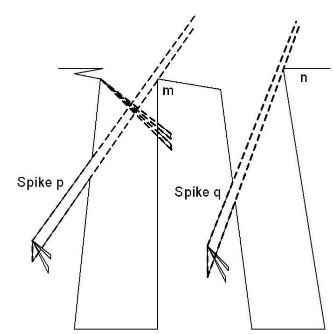

Figure 4: One modified variable pattern

Figure 5 shows that each variable pattern structure has $2l + 1$ "fork-shaped" gadgets in it (when $k = 3$ and $l = 3$).

The overall structure is shown in Figure 6, for the corresponding 3-SAT formula $(u_1 + u_2 + u_3)(u_1 + \bar{u_2} + u_3)(u_1 + \bar{u_2} + \bar{u_3})$.

The whole polygon has $28k + 5n + 2$ distinguished points so that a minimum vertex guard set uses $25k + 4n + 2$ of them as guard locations. It is a minimum vertex guard set if and only if the corresponding 3-CNF formula is satisfiable; check Figure 6 for an example. □

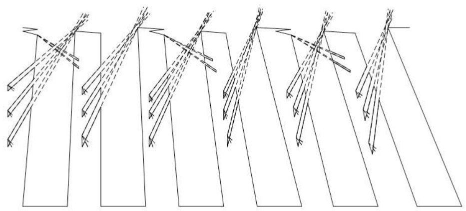

Figure 5: Three modified variable patterns

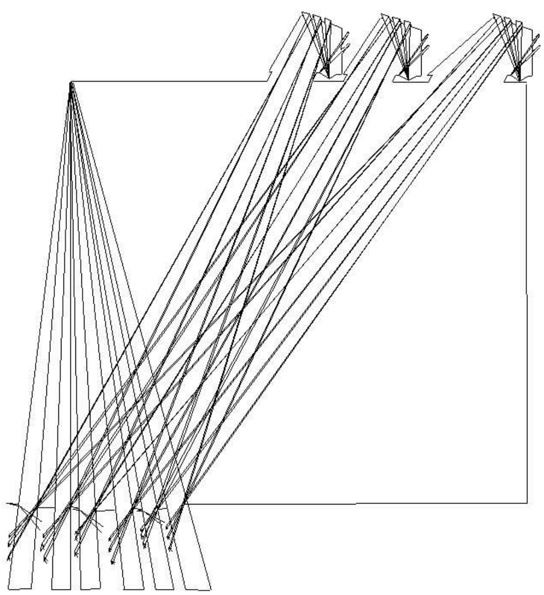

Figure 6: Modified structure for reduction from 3-SAT to two-guard art gallery problem. The corresponding 3-SAT formula is $(u_1+u_2+u_3)(u_1+\bar{u_2}+u_3)(u_1+\bar{u_2}+\bar{u_3})$

3 Approximation Algorithm

Since finding an optimum solution for the two-guard art gallery problem is NP-hard, we turn our attention to finding a polynomial-time approximation algorithm for the problem. Our approach is similar to Ghosh's approximation algorithm [5] for minimum vertex guard for the traditional art gallery problem. We reduce the two-guard problem to a modified minimum set cover problem, which is then solved by a greedy algorithm [6].

Lemma 2 *Any simple polygon can be partitioned into convex parts in polynomial time such that each part is completely visible or completely invisible to each vertex.*

Proof. Given a polygon with n vertices, the most straightforward method to generate such a partition is to draw lines through any pair of vertices of the polygon. The total number of such lines is $O(n^2)$. Each line can intersect at most all other lines. Therefore, it creates $O(n^4)$ convex parts. Since we use lines to partition the interior of polygon, the resulting parts must be convex. It can also be easily proven by contradiction that each part is completely visible or invisible to each vertex. □

Lemma 3 *The problem of finding a minimum vertex guard cover for the two-guard art gallery problem can be reduced to the modified minimum set cover problem in polynomial time.*

Proof. Each part of the partition from Lemma 2 is an element. Each tuple $(v_i, v_j)$is a subset. The elements in each such subset are those convex parts completely visible to both vertices (v_i, v_j). Totally, there are $O(n^2)$ such subsets (tuples). The subsets can be built in polynomial time. The original problem now is reduced to the problem of finding the set of subsets with minimum cardinality of distinct vertices such that the selected subsets contain all elements. □

The approximation algorithm to be presented in this paper is based on Lemma 2 and Lemma 3. The description of the algorithm is as follows.

Step I Partition the whole polygon P into many convex parts $\{p_1, p_2, p_3...p_s\}$(by the method mentioned in the proof of Lemma 2).

Step II For each convex part p_i, which is completely visible or invisible to each vertex, record the list of visible vertices.

Step III Make a set of tuples $\{T_1, T_2, ...T_m\}$ of vertices. In each tuple (v_i, v_j), attach a list of all parts $\{p_1, p_2, ...p_k\}$ such that both v_i, and v_j can cover them. The total number of tuples is $O(n^2)$.

Step IV Apply the modified greedy set cover algorithm. [6]

- **a)** Select the tuple, which can cover the largest number of convex parts.
- **b)** Select each tuple with no new vertex. Then, select the tuple, which can cover the largest number of uncovered convex parts. The selection here depends on the cardinality of new vertices in the tuples: if the tuple contains only one new vertex, then its covering number will be multiplied by two. Otherwise, we keep the same covering number.

c) Run step b recursively until no more uncovered convex parts are left.

Step V Extract distinct vertices from selected tuples. This is the set of vertex guards in the solution.

Theorem 4 *There exists a polynomial approximation algorithm which can solve the minimum vertex two-guard art gallery problem with approximation bound of* $O(\log n)$

Proof. The algorithm we present contains four major steps: partioning the polygon, reducing to a modified minimum set cover problem, running a greedy algorithm for the modified minimum set cover problem, and extracting a set of vertices (as guards). Since the proof for the polynomial running time in each step is not complicated, we leave out the details.

It is known that the greedy set cover algorithm can approximate the minimum set cover problem with an approximation ratio which is logarithmic in the number of elements of the minimum set cover instance [6]. By using the convex parts from the partition as elements, and tuples as subsets, we can modify the proof to show that the approximation ratio is still logarithmic for our algorithm, because we have $O(n^4)$ number of elements in the instance, which is polynomial. Therefore, a polynomial time approximation algorithm with approximation ratio bound of $O(\log n)$ exists. □

4 Collinear Problem

In real applications, we may encounter the collinear problem, which is defined as the follows: *Two distinct guards cannot locate the object's position if the object is exactly on the line which goes through these two guards.* Figure 7 shows a situation in which the collinear problem arises; both guards can "see" the object, however, they cannot determine its position by angles because both rays will be collinear.

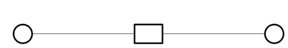

Figure 7: The object is on the line through two guards

However, the approximation algorithm can be modified slightly to deal with this limitation as follows: If a convex part is visible to a tuple (v_i, v_j), but v_i, v_j have collinear problem with one of its edge, delete this convex part from the list of the tuple (v_i, v_j).

5 Conclusion

In this paper, we showed that finding a minimum set of vertex guards for a simple polygon such that every point is seen by at least two guards is NP-hard. We presented a polynomial time approximation algorithm with ratio bound $O(\log n)$. The algorithm reduces this problem to a minimum set cover problem, and then solves it by a greedy approach. We also extended the approximation algorithm to handle situations where coverage by guards that are collinear with the object is not allowed.

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