Hamiltonian Cycles in Triangular Grids

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Abstract

We study the Hamiltonian Cycle problem in graphs induced by subsets of the vertices of the tiling of the plane with equilateral triangles. By analogy with grid graphs we call such graphs *triangular* grid graphs. Following the analogy, we define the class of *solid* triangular grid graphs.

We prove that the Hamiltonian Cycle problem is NP-complete for triangular grid graphs. We show that with the exception of the "Star of David", a *solid* triangular grid graph without cut vertices is always Hamiltonian.

1 Introduction

Grids have proved to be extremely useful in all areas of computer science. Their main usage is as the discrete approximation to a continuous domain or surface. Numerous algorithms in computer graphics, numerical analysis, computational geometry, robotics and other fields are based on grid computations.

Formally, a *square grid*, or *square grid graph* G is induced by a finite subset $\mathbb{G}$ of the infinite integer grid $\mathbb{Z}^2$: the vertices of G are the points in $\mathbb{G}$, the edges of G connect the points of $\mathbb{G}$ that are at unit distance from each other. We will identify a grid graph G with the subset $\mathbb{G}$ that induces the graph.

The infinite grid $\mathbb{Z}^2$ may be viewed as the set of vertices of a tiling of the plane with unit squares. Another plane tiling with regular polygons, the tiling with equilateral triangles, defines an infinite "grid" in the same way; we call this grid infinite *triangular*. A *triangular grid graph* is a graph induced by a finite subset of the infinite triangular grid. We will use the terms triangular (square) *grid graph* and triangular (square) *grid* interchangeably.

As an important special case, the class of "solid" (or, "simple") square grid graphs was introduced [1, 15]. A square grid graph is called *solid* if all of its bounded faces are unit squares.

Grid	Square	Triangular
General	$\mathcal{NPC}$ [14]	$\mathcal{NPC}$, Thm. 1
Degree bounded	deg$\leq$ 3: $\mathcal{NPC}$ [14]	deg$\leq$ 4: $\mathcal{NPC}$, Thm. 2
Solid	$\mathcal{P}$ [15]	$\mathcal{P}$, Thm. 3

Table 1: The hardness of the HCP in grids. $\mathcal{NPC}$ stands for NP-complete, $\mathcal{P}$ stands for polynomial-time solvable.

Related Work

The Hamiltonian Cycle problem (HCP) is one of the basic six NP-complete problems [11]; see also the survey [4]. A lot of effort has been devoted to establishing the boundary between the classes of graphs for which the problem remains NP-complete, and the classes for which it is polynomially solvable. In particular, Dillencourt [7] showed that the problem is NP-complete even if restricted to Delaunay triangulations. Arkin et al. [2] considered Hamiltonian cycles in the duals of triangulations. Cimikowski [5, 6] studied the HCP in inner-triangulations graphs; Dogrusoz and Krishnamoorthy in an unpublished manuscript [8] proved that the HCP for such graphs is NP-complete.

The HCP in square grid graphs has been the subject of extensive research [12, 14, 13, 10, 9, 15, 1]. In general, the problem is NP-complete [12, 14, 13]. It was proved that in solid square grids the HCP is polynomial [15]. It was shown [1] that in a solid square grid on N vertices there exists a tour of length at most $6N/5$ visiting all grid vertices; such a tour can be computed in linear time. Table 1 summarizes known results on the HCP in grid graphs.

Our Contributions

To the best of our knowledge, the HCP in triangular grids has not been considered previously. In this paper we provide an algorithmic study of the problem and give some related results. We prove that the HCP in triangular grid graphs is NP-complete; further, we show that the problem remains NP-complete even if restricted to grids with maximum degree 4. We extend the notions of "solid" to triangular grids; we show that, except for one counterexample, any solid triangular grid without

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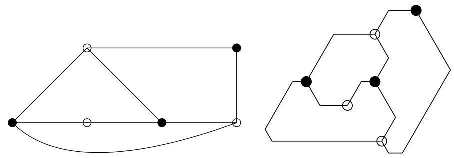

Figure 1: G' and the embedding.

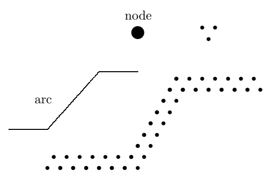

Figure 2: The gadgets.

cut vertices is Hamiltonian and that the Hamiltonian cycle in it can be found in linear time. This result has a couple of straightforward implications, which we discuss in Section 3.

2 Triangular Grid Graphs

Itai et al. [12] and Johnson and Papadimitriou [13] proved that the HCP in square grid graphs is NP-complete by a reduction from HCP in undirected planar bipartite graphs with maximum degree 3. We follow the idea of [14] to show that the HCP in triangular grids is NP-complete.

Let G' be an undirected planar bipartite graph with maximum degree 3; the nodes are 2-colored "black" and "white". We say that G' has *nodes* and *arcs* saving the terms *vertices* and *edges* for the triangular grid graph G that we build from G' as follows. First, G' is embedded in the plane, with the arcs drawn by paths going at 0, 60 or 120 degrees to the x-axis, so that the turn angles are 120^o at each corner along an embedded polygonal arc (Fig. 1). The embedding is then represented by a triangular grid graph G with nodes and arcs simulated by the gadgets shown in Figure 2.

In detail, the nodes are represented by the unit triangles; the arcs are simulated by "tentacles". The triangles corresponding to the black (resp., white) nodes of G' are called black (resp., white). A tentacle-arc is connected to the black triangle with a "pin" connection (Fig. 3, left) and to the white triangle with an "arm" connection (Fig. 3, right).

The only means of traversing a tentacle is either by a *return* path (Fig. 4, left) or by a (kind of a) *cross* path (Fig. 4, right). Of course, there may be many different cross paths, but the essential difference between the return and the cross paths is that the former connects the tentacle vertices aligned along a line, while the latter "jumps back and forth" between the two lines that

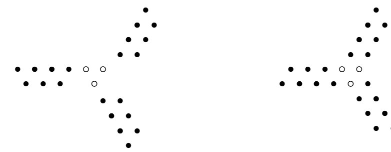

Figure 3: A "pin" connection (left) and an "arm" connection (right). The node gadgets are shown with hollow circles.

Figure 4: The paths.

bound the tentacle. The idea of the difference is that a cross path connects the two node gadgets at its ends, while a return path just traverses the vertices in the tentacle, returning to the same end from which it started.

Theorem 1 *The HCP for triangular grid graphs is NP-complete.*

Proof. (Sketch) If G' has a Hamiltonian cycle, then G has one, which traverses the black and white triangles of G in the order of the corresponding nodes of G' in the cycle. It traverses by cross paths the tentacles that correspond to arcs in the cycle. The remaining tentacles are picked up by return paths from the adjacent white triangles.

Conversely, any Hamiltonian cycle $\mathcal{C}$ of G comes from a Hamiltonian cycle of G' in this way. Indeed, it is not hard to see, by inspection of Fig. 3, that in $\mathcal{C}$ any triangle is attached to exactly two cross paths. □

Triangular Grids of Maximum Degree 4

Papadimitriou and Vazirani [14] also proved that the HCP in square grid graphs is NP-complete even when restricted to graphs of maximum degree 3; Buro [3] gave an alternative proof. In this section we prove that the HCP in triangular grids is NP-complete even when restricted to grids of maximum degree 4.

The graph G constructed in the proof of Theorem 1 has certain vertices of degree 5, namely, the vertices of the white triangles and the inner points of the angles of the tentacles. Figure 5 shows how the construction may be modified so that the resulting graph has vertices of degree 4 or less.

Theorem 2 *The HCP for triangular grid graphs with maximum degree 4 is NP-complete.*

Hamiltonicity of Solid Triangular Grids

As in the case of square grids, we say that a triangular grid is *solid* if every bounded face of it is a unit equilateral triangle. In [1], a linear-time algorithm is proposed

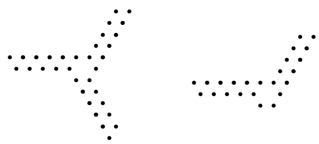

Figure 5: Left: Modified white triangle. Right: Modified turn of a tentacle.

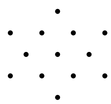

Figure 6: The only non-Hamiltonian solid triangular grid graph: the Star of David.

to find a cycle of length at most $6N/5$ through any solid square grid graph on N vertices. The algorithm takes the cycle around the boundary of (the unbounded face of) the graph and attaches to it all of the internal vertices at a low cost. In this section we show that the algorithm can be extended to the case of solid *triangular* grids. Following [1], we only consider graphs without *cut vertices* (a cut vertex is a vertex whose removal disconnects the graph). It appears that the connectivity of a triangular grid is so high that, with the exception of one particular graph (which we call the "Star of David", Fig. 6), all solid triangular grids without cut vertices are Hamiltonian; in fact, our algorithm will constructively find a Hamiltonian cycle in any solid triangular grid in time linear in the size of the description of the grid.

Let $G = (V, E)$ be a solid triangular grid graph without cut vertices; let C be the cycle going through the vertices of G that are on the boundary of its unbounded face; since there is no cut vertex in G, C is simple. We argue that C can be modified, through a sequence of local modifications, into a cycle that visits all vertices of G. Let C' denote the tour at any particular stage of the modification; we maintain the invariant that C' is a simple cycle within G such that all of the vertices of G that have not been visited by C', $V \setminus C'$, are inside C'. Our modifications are "monotone" in that each modification will result in C' visiting a superset of the vertices that it previously visited.

Before stating our algorithm, we argue that without loss of generality it suffices to consider only the grids without "bottlenecks." Following [1], we say that two edges $\{ac, bd\}$ of C form a *bottleneck* if (1) they are the opposite sides of a unit rhombus, contained within C; and (2) neither of the other two sides of the rhombus (we call each of these sides a *cork*) is an edge of C (Fig. 7, left). (Note that an edge of C can participate in two bottlenecks.) To justify the "bottlenecklessness" assumption, we observe that whenever G has a bottleneck, the graph can be cut through the bottleneck(s)

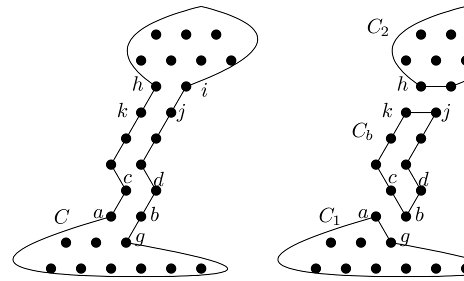

Figure 7: Left: $\{ac, bd\}$, $\{ac, bg\}$, $\{hk, ij\}$ are examples of bottlenecks; $\{ab, cd\}$, $\{ag, bc\}$, $\{hi, kj\}$ are the corresponding cork pairs. Right: G is cut by the bottlenecks $\{ac, bg\}$, $\{hk, ij\}$; C_1, C_2 and C_b use the corks ag, ij, and bc and kj, respectively. As can be seen from Figs. 8 and 9, the modifications *Welcome-1* and *Welcome-2* are "conservative" in that the Hamiltonian cycles through G_1 and G_2, obtained by modifying C_1 and C_2, will still use the corks; it enables splicing the three cycles into a Hamiltonian cycle through G.

into three subgraphs, G_1, G_2 and G_b (with G_b possibly empty), such that (1) neither of G_1, G_2, G_b is a Star of David; (2) each of G_1, G_2 has fewer bottlenecks than G; (3) the cycle C_b around the boundary of G_b is a Hamiltonian cycle through G_b; (4) the cycles C_1, C_2, C_b around the boundaries of G_1, G_2, G_b use the corks of the bottlenecks through which G was cut; and, (5) the modifications *Welcome-1* and *Welcome-2* (see below), applied to each of C_1 and C_2 to obtain the Hamiltonian cycles C'_1 and C'_2 through G_1 and G_2, do not touch the corks (i.e., C'_1 and C'_2 still use the corks). See Fig. 7, right. This means that the cycles C'_1, C'_2, C_b can be spliced into a Hamiltonian cycle through G. Thus, by induction on the number of the bottlenecks, it suffices to consider only the case when G is "bottleneckless."

Theorem 3 *Let G be a connected solid triangular grid graph without cut vertices. Then, unless G is the Star of David, it is Hamiltonian.*

Proof. We consider two types of modifications to C, which we call *Welcome-1* and *Welcome-2*. As mentioned before, we let C' denote the cycle at any particular stage of the modification. For $i = 1, 2$, *Welcome-i* adds i new vertices to C'. *Welcome-2* is applied only when *Welcome-1* cannot be applied.

Welcome-1 is applied as long as there exists an edge ab of C such that the unit equilateral triangle abc has $c \in V \setminus C'$; the modification consists of adding c to C' (Fig. 8). Observe that *Welcome-1* does not create a bottleneck in C'.

Suppose that at some stage *Welcome-1* cannot be applied. Then either C' is already a Hamiltonian cycle in G, in which case we are done, or there exists a vertex

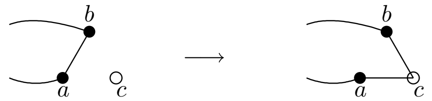

Figure 8: *Welcome-1*: $C' \leftarrow C' \setminus ab \cup ac \cup bc$.

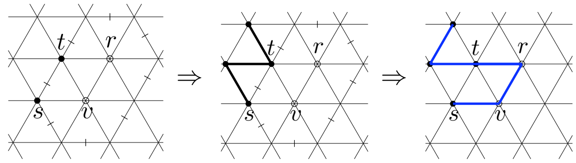

Figure 9: Left: $vr \in V \setminus C'$, $s, t \in C'$. None of the crossed edges may be in C'. Center: the edges of C' that are adjacent to t may be deduced. An edge of C' that is adjacent to s may also be inferred. Right: *Welcome-2* welcomes v and r to C'.

in $V \setminus C'$. Since G is connected, there exists a vertex $v \in V \setminus C'$ such that at least one of the neighbors of v is a vertex of C'. Consider two cases.

All six neighbors of v are in C'. Then, since *Welcome-1* cannot be applied and there is no bottleneck in C', G is a Star of David (see Fig. 6).

At least one of the neighbors of v is in $V \setminus C'$. Let $w \in C'$ be a neighbor of v that is in the cycle C'. At least one of the two vertices, adjacent to both v and w, must be a vertex of C'; otherwise, one of them could have been picked up, by an edge of C' going through w, using *Welcome-1*. Let $x \in C'$ be this neighbor of v and w (obviously, C' does not pass through wx). It is easy to see that then there must exist a unit rhombus $vrst$ such that $vr \in V \setminus C'$, $s, t \in C'$ (possibly, $\{s, t\} \cap \{w, x\} \neq \emptyset$). Since *Welcome-1* cannot be applied, none of the edges of the grid that "surround" v is in C' (Fig. 9, left). This uniquely defines which of the edges adjacent to t are used by C' (Fig. 9, center). Since C' has no bottleneck (indeed, C did not have bottlenecks, and no bottleneck was created by *Welcome-1*'s), it can be deduced that the distance from s to t along C' is 2 (see Fig. 9, center). Finally, a local modification, *Welcome-2*, may be applied to C' in order to add v and r to the cycle (Fig. 9, right).

Observe that, as with *Welcome-1*, *Welcome-2* does not create bottlenecks in C'. This justifies applying the modifications consistently until C' passes through all vertices of G. □

3 Discussion

The proof of Theorem 3 can be turned into a linear-time algorithm for finding a Hamiltonian cycle. To do this it suffices to apply *Welcome-1* to a whole "row" or a "doublerow" of vertices at once.

A simple corollary from Theorem 3 is that the Traveling Salesperson Problem in solid triangular grid graphs is polynomially solvable.

We left open the HCP in triangular grids of degree at most 3 and finding a short covering tour in *any* triangular grid. Another exciting open problem is the HCP in hexagonal grids.

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