Local Overlaps In Special Unfoldings Of Convex Polyhedra

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Abstract

We define a notion of local overlaps in polyhedron unfoldings. We use this concept to construct convex polyhedra for which certain classes of edge unfoldings contain overlaps, thereby negatively resolving some open conjectures. In particular, we construct a convex polyhedron for which every shortest path unfolding contains an overlap. We also present a convex polyhedron for which every steepest edge unfolding contains an overlap. We conclude by analyzing a broad class of unfoldings and again find a convex polyhedron for which they all contain overlaps.

1 Introduction

An *edge unfolding* of a polyhedron is obtained by cutting some edges and unfolding the resulting surface into a connected planar piece. The edges that are cut in this process will form a spanning tree of the vertices called a *cut tree*. The dual of the cut tree is the *adjacency tree*, in which two faces are connected if their common edge is not cut. A *simple* edge unfolding is one that lies in the plane without overlap.

Shephard conjectured that every convex polyhedron has a simple edge unfolding [6]. In attempts to resolve this conjecture, researchers have proposed numerous classes of unfoldings that are conjectured to be simple for convex polyhedra [2, 5].

In this paper we consider a particular type of overlap. In a 2-*local overlap*, there is an edge (v, w) in the unfolding where the overlapping faces are incident with vertices v and w, respectively; see Figure 1(b). We shall develop conditions on an unfolding that imply a 2-local overlap. We then use this result to construct convex polyhedra for which every unfolding in given classes contains an overlap, negatively resolving some open conjectures.

2 2-Local Overlaps

The following Lemma presents a set of conditions that guarantees a 2-local overlap for convex polyhedra.

Lemma 1 *Let P be a convex polyhedron with cut tree C. Suppose $w \in V(P)$ has degree 1 in C, and is adjacent to $v \in V(P)$ in C. Suppose further that there is an unfolding angle ϕ_0 at v bounded by (v, w) with $\phi_0 > \frac{3\pi}{2}$. Then there exists an angle θ_0 that depends on C and ϕ_0 such that the unfolding implied by C will contain a 2-local overlap if the curvature at w is less than θ_0.*

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See Figure 1 for an illustration of the statement of this lemma. The core idea is (informally) that the face incident with w fits tightly into the space around vertex v. Thus, if the curvature at w is small, the face incident with w cannot "swing out" enough to clear the faces incident with v. Note that this unfolding pattern is similar to Schlickenrieder's unfolding of hanging facets [5] and to the unfolding of polyhedral bands studied by Aloupis et al. [1].

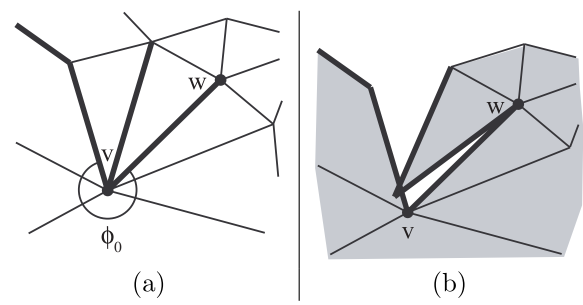

Figure 1: The conditions of Lemma 1. (a) Part of the surface of a polyhedron with cut edges in bold and $\phi_0 > \frac{3\pi}{2}$. (b) The resulting 2-local overlap.

3 Shortest Path Unfoldings

We now apply Lemma 1 to a particular class of unfoldings. Given a polyhedron P and a vertex $v \in V(P)$, the *shortest path tree at v*, $SPT(v)$, is the tree formed by taking the union of the shortest paths from each vertex $w \in V(P)$ to v along the edges of P.

Fukuda made the following conjecture [2]:

Conjecture 1 (Fukuda) *For every convex polyhedron P and every vertex $v \in V(P)$, the cut tree $SPT(v)$ forms a simple unfolding of P.*

Schlickenrieder has already given empirical evidence that this conjecture is false [5]. We disprove this conjecture formally by constructing a counterexample. While the gap filled by this counterexample is admittedly small, it serves as an introduction to the methodology used throughout this paper.

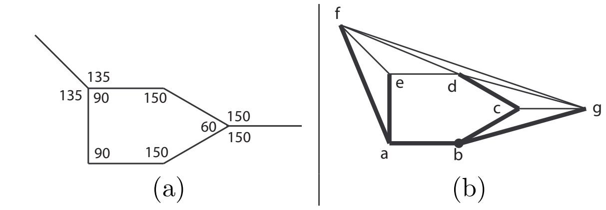

Figure 2: The planar figure used to disprove Conjecture 1. (a) The underlying structure. All line segments are of length 1 and angles are shown in degrees. (b) The completed figure. The bold line segments form $SPT(b)$.

Theorem 2 *There exists a convex polyhedron P with vertex $v \in V(P)$ such that the unfolding corresponding to cut tree $SPT(v)$ contains a 2-local overlap.*

Proof. (Sketch) Consider the graph shown in Figure 2(b). The tree $SPT(b)$ is illustrated in that figure. We can turn this graph into a convex polyhedron by raising vertices c, d, and e off the plane, say by a maximum distance α. This forms a convex terrain, to which we add a bottom face to close the polyhedron. Call this polyhedron $P(\alpha)$.

Note that if α is sufficiently small, then the resulting polyhedron has edge lengths and face angles arbitrarily close to that of the planar figure. In particular, $SPT(b)$ remains the same as in Figure 2(b), and faces (b, c, g) and (c, d, g) together form a component with angle greater than $\frac{3\pi}{2}$ at c. Further, the curvature at d is made arbitrarily small as $\alpha \to 0$.

We therefore conclude that if α is sufficiently small, the conditions of Lemma 1 are satisfied. Thus, for sufficiently small α, cutting $P(\alpha)$ along $SPT(b)$ will create an unfolding with a 2-local overlap. Thus there exists $\alpha > 0$ such that $P(\alpha)$ is the desired counterexample.

□

4 Steepest Edge Unfoldings

We now consider a more complex class of unfoldings, the steepest edge unfoldings. This class of unfoldings was proposed by Schlickenrieder [5].

4.1 Definition

Choose a direction vector ζ. Without loss of generality $\zeta = (0, 0, 1)$ by reorienting space. Then for every vertex v in P, let the *steepest edge for v* be (v, w) such that $\frac{w-v}{|w-v|}$ has maximal z-coordinate. That is, the steepest edge is the edge directed most toward ζ from v. The *steepest edge cut tree* contains the steepest edges of all vertices, except the vertex with maximal z-coordinate. A *steepest edge unfolding* is formed by cutting along a steepest edge cut tree.

Conjecture 2 (Schlickenrieder) *Every convex polyhedron P has a simple steepest edge unfolding.*

This conjecture was motivated by empirical tests, where Schlickenrieder found that a convex polyhedron would unfold without overlap with probability 0.93 when ζ was chosen at random [5]. Nevertheless, we shall construct a polyhedron for which every steepest edge cut tree generates an unfolding with a 2-local overlap.

4.2 Outline

We begin by constructing a convex terrain for which the steepest edge cut tree generates an overlap when ζ lies in some open set. Furthermore, the size of this set is independent of scaling, translation, and rotation of the terrain. We then construct a convex polyhedron by gluing together many copies of this terrain in various orientations. The result will be that for every possible choice of ζ, there is a copy of our terrain that contains an overlap in the corresponding steepest edge unfolding.

4.3 The Terrain

Consider the planar graph M_1 illustrated in Figure 3(a). As before, we can convert this graph into a convex terrain by raising the interior vertices a and b. In particular, given parameter $\alpha > 0$, we denote by $M_1(\alpha)$ the convex terrain formed by raising the vertices a and b to a height of α. Also, as $\alpha \to 0$, the curvatures at a and b become arbitrarily small.

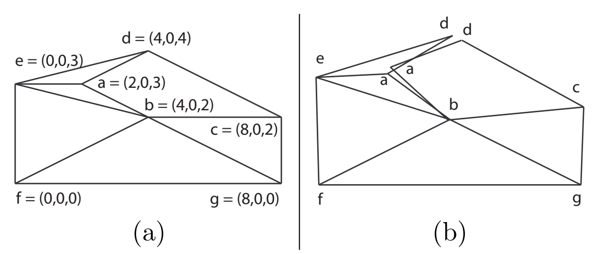

Figure 3: (a) The planar graph M_1. (b) The steepest edge unfolding of $M_1(\alpha)$ for small α and direction vector $\zeta = (0, 0, 1)$.

Lemma 3 *Suppose α is sufficiently small. Then there exists $\phi > 0$ such that if $M_1(\alpha)$ forms part of convex polyhedron P and ζ is a unit vector within an angle of ϕ from vector $\frac{e-f}{|e-f|}$, the steepest edge unfolding of P with direction ζ will contain a 2-local overlap. This is true regardless of any scaling, translation, or rotation of $M_1(\alpha)$.*

Proof. (Sketch) This is very similar to Theorem 2. See Figure 3(b) for an illustration of the steepest edge unfolding of $M_1(\alpha)$ when $\zeta = \frac{e-f}{|e-f|} = (0,0,1)$. This overlap is implied by Lemma 1 (since $\angle dab < \frac{\pi}{2}$ and α is sufficiently small). Since the overlap occurs in the unfolding of $M_1(\alpha)$, it will occur in the unfolding of P as well.

Note that the steepest edge cut tree of $M_1(\alpha)$ remains the same given minute perturbations of the terrain. This observation allows us to introduce ϕ. We describe ζ with respect to e and f, rather than $(0,0,1)$, to make clear the independence of ϕ from rotation and scale of the terrain. □

4.4 The Polyhedron

We now construct our final polyhedron by covering the surface of the sphere with many copies of $M_1(\alpha)$. We can do this in such a way that for any choice of direction ζ there is a corresponding copy of $M_1(\alpha)$ such that $\frac{e-f}{|e-f|}$ is within ϕ of ζ [3]. Then Lemma 3 implies that the steepest edge unfolding with direction ζ contains an overlap. This implies the following theorem.

Theorem 4 *There exists a convex polyhedron P such that every steepest edge unfolding of P contains a 2-local overlap.*

5 Normal Order Unfoldings

We now consider a broad class of unfoldings: the *Normal Order* unfoldings. This class was motivated as a generalization of steepest edge unfoldings. We then construct a polyhedron P such that every normal order unfolding of P contains an overlap.

5.1 Definition

Let P be a convex polyhedron. Choose a direction vector ζ; without loss of generality $\zeta = (0,0,1)$. Given $f \in F(P)$, let n_f be the outward-facing unit normal for f. Denote by $z(f)$ the z-coordinate of n_f. Informally, $z(f)$ is a measure of the "height" of f with respect to P.

Now consider a cut tree C of P, with corresponding adjacency tree A and unfolding P'. We say that the unfolding P' is a *normal order* unfolding if, for all $\delta \in [-1,1]$, the set $\{f \in F(P) : z(f) \leq \delta\}$ is connected in A. In other words, if face f of P does not have minimal $z(f)$ then f must be adjacent to a face g along an edge that is not cut, where $z(g) \leq z(f)$. Any face f that does not satisfy this property is said to be a *dangling face*. For example, for the polyhedron shown in Figure 1(a), if ζ faces the top of the page then the overlapping face incident with w in Figure 1(b) is a dangling face.

5.2 Motivation

Our definition of normal order unfoldings is motivated by the steepest edge unfoldings. In Schlickenrieder's paper, there are examples of complex convex polyhedra with simple steepest edge unfoldings [5]. As an informal intuition, the success of these unfoldings appears to derive from their tendancy to "expand outward" from a central point. See Figure 4. The definition of normal order unfoldings attempts to capture this property, by preventing dangling faces.

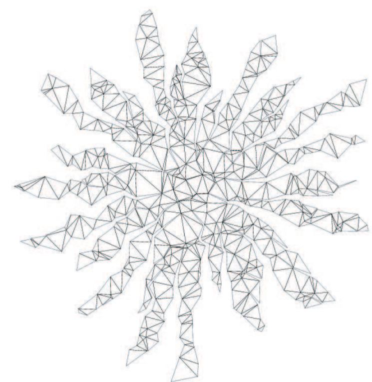

Figure 4: A simple steepest edge unfolding of a convex polyhedron. Image due to Schlickenrieder [5].

Unfortunately, as we shall show, it is not the case that every convex polyhedron has a simple normal order unfolding.

5.3 Construction

We shall now construct a polyhedron for which every normal order unfolding contains an overlap. The method of construction is very similar to that for steepest edge unfoldings. We simply modify the terrain M_1 to generate overlaps for normal order unfoldings.

Consider the planar graph M_2 illustrated in Figure 5(a). The important thing to notice about this graph is that certain angles, illustrated in the figure, are all less than $\frac{\pi}{2}$.

Consider raising the interior vertices of M_2 by at most α in such a way that each face of M_2 remains planar. Call the resulting convex terrain $M_2(\alpha)$. Suppose that vector $(0,0,1)$ points towards the top of the page in Figure 5(a). Then if we take $\zeta = (0,0,1)$, all normal order unfoldings of $M_2(\alpha)$ are shown in Figure 5. In each case, Lemma 1 applies if α is sufficiently small, so an overlap occurs. The exception is Figure 5(d), where a 3-local overlap occurs. However, the conditions leading to this overlap are very similar to those in Lemma 1, so its existence can be proven without too much additional work [3].

Lemma 5 *Suppose ζ is a unit vector that has angle at most ϕ from $(0,0,1)$. Then any normal order unfolding of $M_2(\alpha)$ with respect to ζ contains an overlap, when α and ϕ are sufficiently small.*

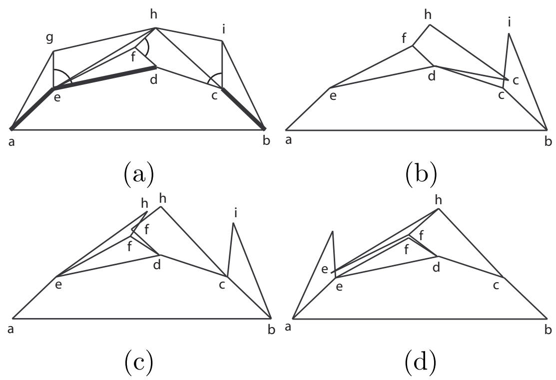

Figure 5: (a) The planar graph M_2. The marked edges are *not* cut in a normal order unfolding, and the marked angles are less than $\frac{\pi}{2}$. (b,c,d) Portions of the normal order unfoldings of $M_2(\alpha)$

We can now construct a convex polyhedron by gluing together many copies of $M_2(\alpha)$, in the same way as for Theorem 4. This implies the following theorem.

Theorem 6 *There exists a convex polyhedron P such that every normal order unfolding of P contains an overlap.*

A simple generalization of Theorem 6 yields the following property for any algorithm that would positively resolve Shephard's conjecture.

Theorem 7 *Any unfolding algorithm that generates simple unfoldings for all convex polyhedra must allow an arbitrary number of dangling faces.*

Proof. (Sketch) We can increase the number of copies of $M_2(\alpha)$ used in constructing the polyhedron for Theorem 6, so that an arbitrary number of copies will generate overlaps for any chosen direction vector ζ. □

6 Conclusion

We have developed a methodology for constructing convex polyhedra for which a given class of unfoldings contains no simple unfoldings. This was used to negatively resolve conjectures by Fukuda and Schlickenrieder. Further, we applied this method to a broad class of unfoldings – the normal order unfoldings – to develop a property of any unfolding algorithm that generates only simple unfoldings for convex polyhedra. This last counterexample also serves to break the intuition that one can always construct a simple unfolding that "expands outwards" monotonically from a point.

This work leaves open a number of questions for future research. First, our definition of normal order unfoldings could be altered in subtle ways. It is possible that a slightly different definition could preserve the informal notion of creating starlike unfoldings, yet not fall to the type of counterexample presented in this paper.

Also, Lemma 1 gives only a set of sufficient conditions for a limited type of overlap. We could strengthen our ability to construct counterexamples by considering the more general notion of k-local overlaps [3]. One could imagine generalizing Lemma 1 to a full characterization of necessary and sufficient conditions leading to k-local overlaps. Such a characterization would be a powerful tool and significant progress towards resolving Shephard's conjecture. However, even extending Lemma 1 to 3-local overlaps could prove enlightening in the study of convex polyhedra unfoldings.

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