# Simple Characterization of Polygons Searchable by 1-Searcher

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## Abstract

Suppose intruders are in a dark polygonal room and they can move arbitrarily fast, trying to avoid detection. A *boundary 1-searcher* can move along the polygon boundary, equipped with a flash light that she can direct in any direction. A polygon is *searchable* if there is a schedule for the searcher in order to detect the intruders no matter how they move. We identify three simple *forbidden patterns* such that a given polygon is searchable by a boundary 1-searcher if and only if it has none of them. The concept of *sweeping* the visibility diagram greatly facilitates the proof.


## 1 Introduction

Suzuki and Yamashita defined a *k-searcher* equipped with $k$ flashlights who tries to detect intruders who can move faster than the scanning speed of flashlights by eventually illuminating them [7]. For recent results on this topic, the reader is referred to [1, 2]. Two types of visibility, *grazing* and *non-grazing* visibility, have been defined in the literature [3, 7]. Two points $a, b$ in a simple polygon $P$ are said to be *mutually visible* under grazing (resp. non-grazing) visibility, if $\overline{ab} \subseteq P$ (resp. $\overline{ab} \subseteq P \setminus \partial P$), where $\partial P$ denotes the boundary or $P$. With grazing visibility a 1-searcher need not stay on the boundary all the time. In [4] and [8], respectively, Park et al. and Tan worked on the characterization of polygons that are searchable by a general 1-searcher with grazing visibility. Tan tried to characterize the polygons that are searchable by a *boundary* 1-searcher [9]. Unfortunately, the proofs of correctness in these papers are several pages long and difficult to follow, and the main results in [8] and [9] contain errors.

The aim of this paper is to characterize the simple polygons that are searchable by a boundary 1-searcher with grazing visibility, in terms of a new set of three simple forbidden patterns. One of them is a special case of the pattern given in [7]. The other two resemble those in [9], but the errors have been corrected. Most importantly, we use a high-level tool, called the *V-diagram* developed in [2], to make the proof of correctness short (less than 3/4 of a page, in contrast to more than 7 pages in [9]). Only minor modifications are necessary to deal with non-grazing visibility.


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## 2 Preliminaries

### 2.1 Notation

A (simple) polygon $P$ consists of $n$ ($\geq 3$) vertices and $n$ edges connecting adjacent vertices. The vertices preceding and succeeding $v$ in the clockwise order are denoted by $Pred(v)$ and $Succ(v)$, respectively. A *reflex vertex* is one whose interior angle is larger than $180^o$. For a reflex vertex $v$, let $F(v)$ (resp. $B(v)$) denote the point on $\partial P$ where the extension of the edge $(Pred(v), v)$ (resp. $(Succ(v), v)$) towards the interior of $P$ exits $P$. Let $a, b \in \partial P$. The clockwise section of $\partial P$ from point $a$ to point $b$ is denoted by $\partial P[a, b]$. If $c \in \partial P[a, b] \setminus \{a, b\}$, we write $a \prec c \prec b$. For a reflex vertex $v$, the section $\partial P[v, B(v)]$ (resp. $\partial P[F(v), v]$) is called the *clockwise* (*cw*) (resp. *counterclockwise* (*ccw*)) *component* associated with $v$.

We fix an arbitrary point on $\partial P$ as the origin, and measure all distances along $\partial P$ clockwise from the origin. Let $|\partial P|$ denote the length of $\partial P$. For $x \in \mathbb{R}$ (the set of all real numbers), $x$ represents the point on $\partial P$ that is at distance $x - k|\partial P|$ from the origin, where $k$ is an integer such that $0 \leq x - k|\partial P| < |\partial P|$. We thus can consider $x$ also as a point on $\partial P$. Note that there are infinitely many real numbers $x \in \mathbb{R}$ that represent a single point on $\partial P$.

### 2.2 Visibility diagram and search path

Let $x, y \in \mathbb{R}$. The *visibility space*, denoted by $\mathcal{V}$, consists of the infinite area between and including the lines $y = x$ (*start line S*) and $y = x - |\partial P|$ (*goal line G*), as shown in Fig. 1 [2].

The *visibility diagram* (*V-diagram* for short) for a given polygon is drawn in $\mathcal{V}$ by shading some areas in it gray as follows: point $(x, y) \in \mathcal{V}$ is gray if points $x$ and $y$ are not mutually visible. We use $x$ ($y$) to represent the searcher (beam head) position. Note that in Fig. 1 point $p$ is gray if and only if points $p'$ and $p''$ are also gray. Each reflex vertex $r$ gives rise to two shaded areas in each section of length $|\partial P|$ in the V-diagram, as shown in an idealized form in Fig. 2.

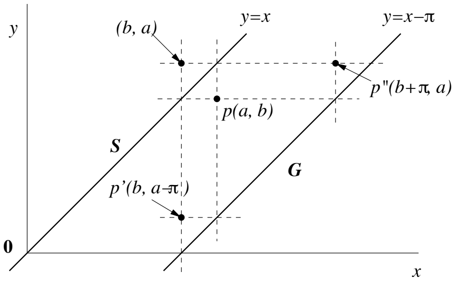


Figure 1: Visibility space ($\pi = |\partial P|$).

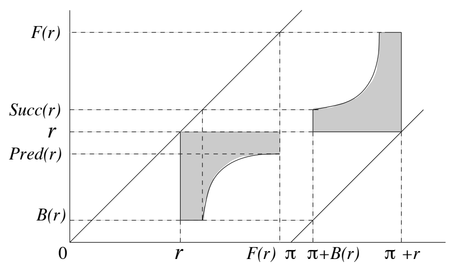


Figure 2: SE and NW barriers due to reflex vertex $r$.

We call each such shape a *barrier* and a barrier whose corner is touching line $S$ or $G$ is called a *southeast* (*SE*) or *northwest* (*NW*) *barrier*, respectively [2]. See Fig. 3 for an example of a polygon and its V-diagram. (Imagine that the blackened area is gray.) In a *skeleton V-diagram* to be used in Sec. 4, each barrier is shrunk to a horizontal and vertical line segments without losing the topological information [2].

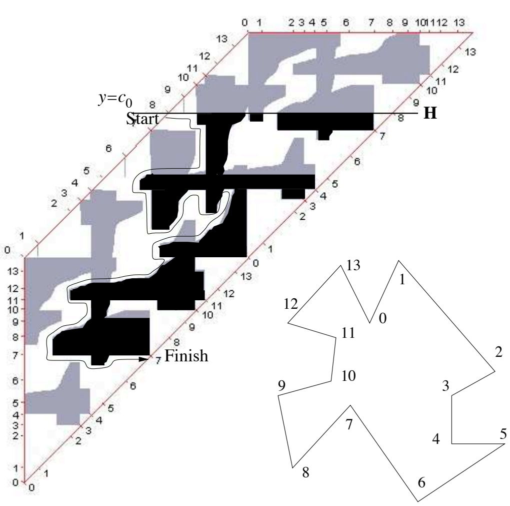


Figure 3: An example polygon and its V-diagram.

A *search path* is a path from $S$ to $G$ that stays within white areas, except that it may cross gray areas from right to left, e.g., the curve from Start to Finish in Fig. 3. Crossing a gray area from right to left reflects the movement of the beam head from the current position to a nearer reflex vertex as the beam is moved to the left [2, 3].

**Theorem 1** [2] *A given polygon is searchable by a boundary 1-searcher if and only if there is a search path in its V-diagram.* □

## 3 Searchability test by sweeping D-diagram

It is known that a polygon is searchable by a boundary one-way $\infty$-searcher (who has $360^o$ vision) if and only if it is searchable by a boundary 1-searcher [6]. In terms of Fig. 3, let us see how an $\infty$-searcher can clear a polygon. Suppose she is initially placed at an arbitrary position $c_0 \in \partial P$, where $c_0 \in \mathbb{R}$. In the V-diagram, let $H$ denote the intersection of line $y = c_0$ and the gray region. Then $H$ corresponds to the parts of $\partial P$ that the $\infty$-searcher cannot see from point $c_0$.

The counter-clockwise movement of the $\infty$-searcher can be simulated by the downward movement of the line $y = c$, as $c$ is decreased from $c_0$. Let $X$ be a maximal contiguous segment of $H$. Such an $X$ may grow or shrink as $c$ is decreased. If $X$ shrinks to the empty set, it implies that part of the boundary has been cleared by the $\infty$-searcher. Suppose a new section is added to $H$ as $c$ is decreased. This is due to the fact that a previously clear section has become invisible because it became hidden behind some reflex corner. Therefore, this new section is still clear, although it is currently invisible.

**Sweeping Rule:** Draw a horizontal line $y = c_0$ across the V-diagram at an arbitrary position. If it doesn't cross the gray region, stop. Otherwise, let $H$ denote the intersection between $y = c_0$ and the gray region. Paint each point $p$ in the gray region black, if $p$ can be reached from a point in $H$ by a path that moves left, right or downward without leaving the gray region. □

To visualize the sweeping operation, imagine that $H$ emits black fluid that flows downward using the gray region of the V-diagram as a conduit. The fluid flows left, right or downwards (but not upwards) constrained by the conduit, making the parts of the conduit it comes in contact black.

Suppose that the painting by the sweeping operation terminates as in Fig. 3. We show later that whether it terminates or not is independent of the choice of $c_0$. A search path can be constructed by following the boundary of the painted areas, starting just below line $y = c_0$ at the start line $S$. See the path from Start to Finish shown in Fig. 3. The line $y = c_0$ is considered black. It follows from results in Sec. 5 of [6] that

**Theorem 2** *Polygon $P$ is searchable by a boundary 1-searcher if and only if the area of its V-diagram that is painted black by the sweeping operation is finite.* □

## 4 Characterization

We first identify some patterns to be used to define our three forbidden patterns. Two reflex vertices $u$ and $v$ such that neither of them is in the cw component of the other (i.e., $B(u) \prec v \prec B(v) \prec u$ holds) form a *cw non-dominating pair* (NDPcw, for short). For example, see $u$ and $v$ in Fig. 4(a). Similarly, two reflex vertices $u$ and $v$ such that neither of them is in the ccw component of the other (i.e., $u \prec F(v) \prec v \prec F(u)$ holds) form a *ccw non-dominating pair* (NDPccw, for short). For example, see $u$ and $v$ in Fig. 4(b). Finally, two reflex vertices $u$ and $v$ such that $u$ ($v$) is not in the ccw (cw) component of $v$ ($u$) (i.e., $u \prec \{B(u), F(v)\} \prec v$ holds) form a *symmetric NDP*.[1] For example, see $u$ and $v$ in Fig. 4(c). Vertices $u$ and $v$ need not be mutually visible in Fig. 4.

Note that the barriers of an NDP intersect, as seen in the skeleton V-diagrams in Fig. 5.

An *NDPcw couple* (*2NDPcw* for short) consists of three reflex vertices, $u, v$, and $w$ such that both $\{u, v\}$ and $\{u, w\}$ form NDPcws. For example, see Fig. 4(a). An *NDPccw couple* (*2NDPccw* for short) is shown in Fig. 4(b). Pattern $\tau$ is obtained by combining an NDPcw or NDPccw with a symmetric NDP that share reflex vertex $u$. For example, see Fig. 4(c) & (d). Their V-diagrams are shown in Fig. 5. We can see

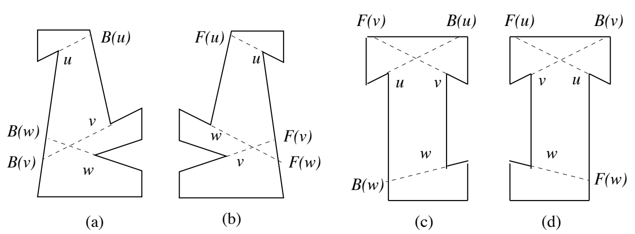


Figure 4: (a) 2NDPcw; (b) 2NDPccw; (c)(d) Pattern $\tau$.

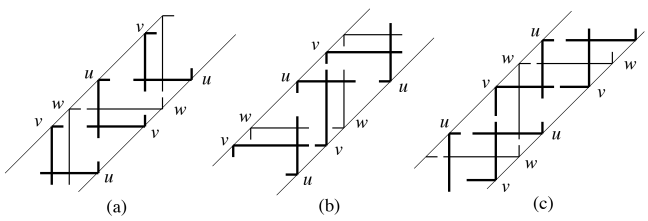


Figure 5: Skeleton V-diagrams for: (a) 2NDPcw; (b) 2NDPccw; (c) Pattern $\tau$ in Fig. 4(c).

from Fig. 5(c) that any polygon that contains the $\tau$ pattern is not searchable [7], since sweeping does not terminate.[2] In other words, pattern $\tau$ is *forbidden*. Each edge in $\partial P[v, w]$ in Fig. 4 (a) (resp. (b)) is said to be *covered* by the 2NDPcw (resp. 2NDPccw). If $u \prec \{B(u), F(v)\} \prec v$ (i.e., $\{u, v\}$ form a *symmetric NDP*), each edge in $\partial P[v, u]$ is said to be *covered* by *trap* (or *deadlock*) $\{u, v\}$. Refer to Fig. 5(c) to see where trap $\{u, v\}$ is located in the V-diagram.

We now introduce three conditions for a given polygon $P$ to be *not* searchable by a boundary 1-searcher.

- FP1: $P$ contains pattern $\tau$.
- FP2: Each edge of $P$ is covered by a trap or 2NDPcw.
- FP3: Each edge of $P$ is covered by a trap or 2NDPccw.

Note that pattern $\tau$ is a special case of the forbidden pattern that was identified in [7] and cited in [9] as condition *C1*. FP2 and FP3 are the corrected versions of *C2* and *C3* in [9], respectively.[3]

**Theorem 3** *A given polygon $P$ is searchable by a boundary 1-searcher if and only if it satisfies none of the three conditions FP1, FP2 or FP3.*

**Proof.** *Necessity:* Let us apply the sweeping operation to the V-diagram of polygon $P$ satisfying any of these conditions. If $P$ contains the $\tau$ pattern, the black fluid keeps flowing, i.e., the area that gets blackened will be infinite. See Fig. 5(c). Similarly, if FP2 or FP3 holds, then black fluid keeps flowing and is never blocked. The necessity thus follows from Theorem 2.

*Sufficiency:* Assume that the given polygon $P$ satisfies none of the conditions, FP1, FP2, or FP3. We want to show that the parts of the gray region that are painted black are finite regardless of where we draw the initial sweep line $y = c_0$.

Let $e$ be an edge of $P$ that is covered by neither a trap nor a 2NDPccw, and let $d$ be an edge of $P$ that is covered by neither a trap nor a 2NDPcw. Such $e$ and $d$ always exist by our assumption. In the V-diagram, both $e$ and $d$ appear infinitely many times. We pay attention to one of their instances such that $d + |\partial P| < e$ in the V-diagram. We want to show that painting due to the sweeping operation eventually stops, i.e., the blackened sections of line $y = c$ vanish as $c$ is decreased. Consider any SE barrier $S'$ that $y = c$ encounters after sweeping past $e$. We claim that the intersection of $y = c$ and $S'$ does not get black *when $S'$ is first encountered* by the line $y = c$ as it moves down, and moreover, there is no SE barrier $S''$ lying below $S'$ that gets black when it is first encountered. The reason is as follows: (i) If black fluid is passed to $S'$ via an SE barrier that touches the

[1] Also called a *BF-pair* [9].

[2] The V-diagram for Fig. 4(d) is "symmetric" to Fig. 5(c).

[3] The second condition for defining *C2*, for example, is $v_2 < Forw(v_1) < v_3$ in the original notation of [9], which resembles $v_3 \prec B(v_1) \prec v_2$ in our notation, but *Forw*() does not reflect grazing visibility. It should be replaced by the condition that $v_1$ and $v_2$ form an NDPcw, or $v_1 \prec B(v_1) \prec v_2$. Similarly for condition *C3*.

start line $S$ above $S'$, it implies $e$ is covered by a trap, a contradiction. (See Fig. 6(a).) (ii) Assume that black fluid is passed to $S'$ via an NW barrier $N$ which intersects $S'$. Such an $N$ cannot intersect a blackened SE barrier above $e$, since $e$ is not covered by any 2NDPccw. (See Fig. 6(b).) (iii) So, suppose such an $N$ intersects a blackened NW barrier. If $S'$ intersects another SE barrier $S''$, as shown in Fig. 6 (c) and (d), it implies the existence of the $\tau$ pattern, a contradiction.

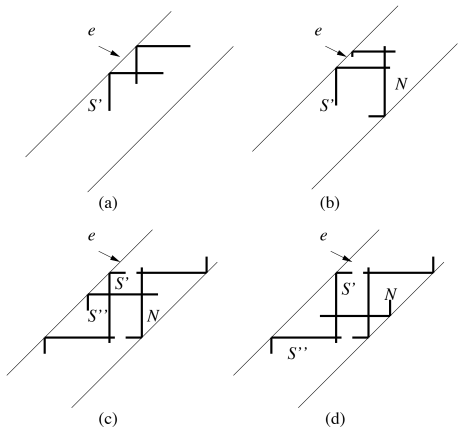


Figure 6: Illustration for the proof of Theorem 3.

It follows that there is no SE barrier lying below edge $e$ that becomes black at the point where it touches the start line $S$. (A part of such an SE barrier may still be blackened, as we see below.) Therefore, black fluid is propagated from an NW barrier to another NW barrier.

Assume now that NW barrier $N$ that touches line $G$ at a point below edge $d$ is still blackened. There are two possibilities. (a) Black fluid was passed to $N$ from

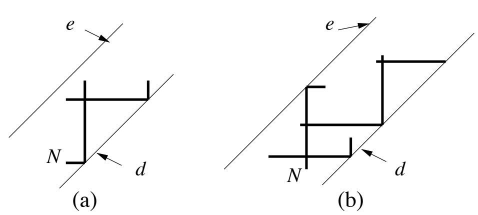


Figure 7: More illustration for the proof of Theorem 3.

an NW barrier lying above $d$, as in Fig. 7(a). This is impossible, since this implies that $d$ is covered by a trap. (b) Black fluid was passed to $N$ from an SE barrier, which in turn was blackened via an NW barrier lying above $d$. This is also impossible, since this implies that $d$ is covered by a 2NDPcw as in Fig. 7(b). It follows that black fluid flow stops after edge $d$. □

## 5 Conclusion and Discussion

We identified three forbidden patterns and proved that the polygon is searchable by a boundary 1-searcher if and only if none of them is present. The proof of correctness has been shortened drastically, compared with past work on similar topics, thanks to high-level concepts of the V-diagram and the sweeping operation. In [4], [5], [8], and [9], for example, sufficiency is proved by showing that a particular searching algorithm is correct, which is a tedious and error-prone process, as evidenced by the length of their proofs and errors in [8] and [9]. Since our objective is a characterization, we could use a higher-level, less problem-specific and more general approach. The time complexity of the test implied by Theorem 3 is $O(n \log n)$. We can deal with the general (non-boundary) 1-searcher with grazing visibility [4], by disallowing the movement of black fluid to the right in the sweeping operation.[4]

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[4]Condition N2 in [4] is not in terms of a simple forbidden pattern, but it is equivalent to FP2 and FP3 with just a 2NDPcw and 2NDPccw, respectively, and no trap. Condition N3 is equivalent to FP2 with just a trap and no 2NDPcw.