Geometric Separator for d-dimensional ball graphs

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Abstract

We study the graph partitioning problem on d-dimensional ball graphs in a geometric way. Let B be a set of balls in d-dimensional Euclidean space with radius ratio δ and λ-precision. We prove that it can be partitioned into three sets B_S, B_I, B_E such that the intersection of B_I and B_E is empty, and for some constant α, the volume of B_I and B_E is less than α portion of volume of B, and the volume of B_S is of size $O(\sigma/\lambda, (\mu(B))^{1-\frac{1}{d}})$, where $\mu(B)$ is the volume of B. We also provide a randomized algorithm to find such a partition in linear time.

1 Introduction

Divide and conquer is a commonly used technique in algorithm design. Generally we can not split a graph G into two disjoint subgraphs for the purpose of divide and conquer. A alternative way is to find a small portion S of G such that by removing S from G, $G - S$ forms two disconnected subgraphs of G, each contains some portion of G. S is smaller than some predefined function f of G(eg. $f(|G|) = \sqrt{8|G|}$). Such S is called a separator of G. Small separator makes the divide and conquer scheme possible on graphs.

Definition 1 (α-split-$f(n)$-separator) *$\mathcal{G}$ is a class of graphs closed under the subgraph relation. $\forall G \in \mathcal{G}$, $|G| = n$, and $0 < \alpha < 1$, if there exists a separator S such that G is separated into three graphs G_S, G_I, G_E such that $|G_S| \leq f(n)$ and $|G_E|, |G_I| \leq \alpha n$. then we call G_S a α-split-$f(n)$-separator of G. Sometimes, we are more interested in the size of the separator, we simply call it $f(n)$-separator.*

There are 2/3-split-$O(\sqrt{n})$-separators [12, 6, 7, 2] and 3/4-split-$O(\sqrt{n})$-separators [15] for planar graphs, $(d+1)/(d+2)$-split-$O(n^{1-1/d})$-separatorsfor meshes. There are also separators in Lebesgue measure [1]. We will discuss these different separators in more detail later.

In this paper, we answer a question raised by Alber and Fiala [1]. For a given set of n disks of bounded radius ratio, Alber and Fiala [1] solved the problem of whether there exists a set of k independent disks in $n^{(O(\sqrt{k}))}$, using their separator theorem in Lebesgue measure on 2-D disk graphs. They challenged the problem of proving the existence of such a separator theorem of Lebesgue measure in 3-D and an algorithm to compute it. We answer the question by proving the existence of a separator theorem of Lebesgue measure in a d-dimensional ball graph, the more general "disk graph", as well as an randomized linear time algorithm to find such a separator.

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2 Previous Works

We discuss different separator theorems in this section. These separator theorems enable us to develop efficient graph algorithms for various graphs [11, 13]. Throughout this paper, all coordinates and distances are defined in Euclidean space. For any two points $\mathbf{x} = \{x_1, \ldots, x_d\}, \mathbf{y} = \{y_1, \ldots, y_d\} \in \mathbb{R}^d$, the distance between $\mathbf{x}, \mathbf{y}$ is defined as $dist(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{d}(x_i - y_i)^2}$.

2.1 Planar Separators

Lipton and Tarjan [12] proved the existence of 2/3-split-$\sqrt{n}$-separator of planar graphs of size n. They also provided a linear time algorithm for finding such a separator. Their separator theorem made the general divide and conquer scheme on planar graphs feasible. The separator for planar graphs was improved in various literatures [7, 6, 2, 15]. The planar separator is applied to the design of planar linear systems [11], in VLSI layouts[9, 10, 17], and in various graph algorithms[12].

2.2 Separators for neighborhood systems

However, not all graphs in practice are planar. Miller, Teng, Thurston, and Vavasis [14, 13] introduced the neighborhood system. A *k-ply neighborhood system* in d-dimension is a collection of closed balls $B = \{B_1, \ldots, B_n\} \in \mathbb{R}^d$ such that no points in $\mathbb{R}^d$ is strictly interior to more than $k+1$ of the balls of B. The following theorem relates the k-ply neighborhood system with the planar graph.

Theorem 2 (Koebe-Andreev-Thurston) *[8, 3, 4, 16] Every planar graph is isomorphic to an intersection graph of a disk packing.*

The intersection graph of a disk packing could be viewed as the 1-ply system in two-dimension. So the class of planar graph is a subset of the class of k-ply neighborhood system in two-dimension, hence a subset of k-ply neighborhood system in d-dimension.

Miller, Teng, Thurston, and Vavasis [13] proved the existence of $\frac{d+1}{d+2}$-split-$(1-1/d)$-separator on k-ply neighborhood system in d-dimension and gave a randomized linear time algorithm in finding such a separator. Their separator could be used in designing algorithms for non-planar graphs and graphs in higher dimension.

2.3 Separators in the Lebesgue measure

The *Lebesgue measure* μ, which measures the volume in Euclidean space, is a commonly used measurement other than the counting measure.

Some properties of the Lebesgue measure includes:

- $\forall S_1, S_2, \mu(S_1) + \mu(S_2) \geq \mu(S_1 \cup S_2)$
- $\forall S_1, S_2, S_1 \subseteq S_2 \rightarrow \mu(S_1) \subseteq \mu(S_2)$

For a set of balls $B = B_1, \ldots, B_n$, $\mu(B) = \mu(\bigcup_{i=1}^{n} B_i)$ is the volume of the union of the balls $B_1, \ldots, B_n$.

The separator theorem for the Lebesgue measure μ is defined similarly.

Definition 3 (α-split-$f(n)$-separator theorem for μ.) *$\mathcal{G}$ is a class of graphs closed under the subgraph relation. $\forall G \in \mathcal{G}$, $|G| = n$, and $0 < \alpha < 1$, if the vertices of G can be partitioned into three sets I, E, S such that there is no edge joins a vertex of I to E, $\mu(I), \mu(E) \leq \alpha\mu(G)$, and $\mu(S) \leq f(\mu(G))$, then such a separator is called a α-split-$f(n)$-separator for the Lebesgue measure μ, or $f(n)$-separator in short.*

Alber and Fiala [1] gave a separator theorem for the Lebesgue measure on disk graphs. In short, a disk graph is a collection of disks in 2D space.

Theorem 4 (Alber and Fiala) *$\mathcal{G}$ is the class of disk graphs with bounded radius ratio σ. There exists constant $\alpha < 1$ and β such that for every graph $G \in \mathcal{G}$, there are three sets $G_I, G_E, G_S \subseteq G$ such that (G_I, G_E, G_S) is a separation of G, satisfying*

1. $\mu(G_S) \leq \sigma^2\beta\sqrt{\mu(G)}$,
2. $\mu(G_I), \mu(G_E) \leq \alpha(G)$.

Such a separator can be found in polynomial time.

Given a disk set of size n in 2D, Alber and Fiala [1] proved that the problem of whether there exists a set of k independent disks is fixed parameter tractable in $2^{O(\sqrt{k})} + n^{O(1)}$ using Theorem 4.

We prove the existence of similar separators for Lebesgue measure in higher dimensional ball graphs, which is the disk graph in higher dimension. Our separator theorem answers Alber and Fiala's question in a more general way.

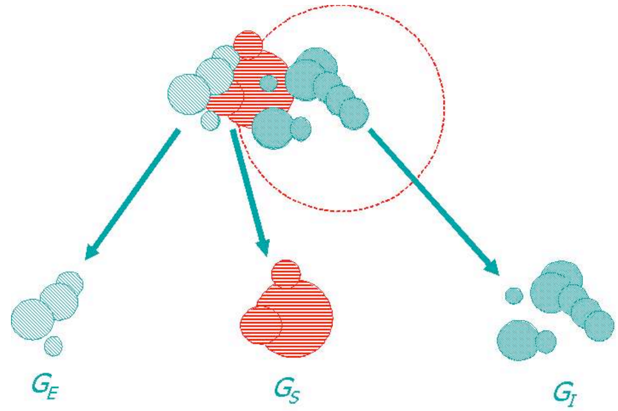

Figure 1: Geometric Separator

3 Geometric Separators for d-Dimensional ball graphs

We need some notations before we proceed with the separator theorem for Lebesgue measure in d-dimensional ball graphs.

Definition 5 (Ball Graphs) *A ball B in d-dimension is defined as $\{b_1, b_2, \ldots, b_d, r\}$, where $r > 0$ is the radius of the ball, and $\{b_1, b_2, \ldots, b_d\}$ are the coordinates of the center of the ball in the d-dimensional Euclidean space. The graph class of* ball graphs*, denoted by $\mathcal{B}$, is the set of all graphs, for which we find a collection of balls $B = \{B_1, \ldots, B_n\}$ such that $G = G_B$.*

The class of ball graphs of *bounded radius ratio* σ, $\mathcal{B}_\sigma$, is a subclass of $\mathcal{B}$ which admit a representation $B = \{B_1, \ldots, B_n\}$ such that the ratio of maximum radius over minimum radius of B is upper bounded by σ. A ball graph B is said to be λ*-precision* if any two center of balls of B is at least λ apart.

Definition 6 (Grid Graphs) *The* grid graph *$\mathcal{H}$ in d-dimension is an infinite undirected graph. Its vertices are defined as all points in the space with all coordinates being integer, There are edges with any two vertices of distance 1. $\forall \mathbf{x_1}, \mathbf{x_2} \in V_{\mathcal{H}}$, if $d(\mathbf{x_1}, \mathbf{x_2}) = 1$, then $(\mathbf{x_1}, \mathbf{x_2}) \in E_{\mathcal{H}}$.*

We define a δ*-grid graph* $\mathcal{H}^\delta$ as δ scaled grid graph $\mathcal{H}$. For a point $p \in \mathcal{H}^\delta$, we say a δ-hypercube is p's δ-hypercube, if p is one of its nodes.

For a collection of balls in d-dimension $B = \{B_1, \ldots, B_n\} \in \mathcal{R}^d$ and a constant $\delta > 0$, we denote W_B^δ the set of all grid points $p \in \mathcal{H}^\delta$ such that at least one of p's δ-hypercube intersects with B. The *covering grid (of span δ) for B*, denoted by H_B^δ, is defined as the graph such that the vertex set is W_B^δ and edges are edges of $\mathcal{H}^\delta$ whose both vertices also lie in W_B^δ.

The following two lemmas reveals the relationship between $k-$ply neighborhood systems and balls of bounded radius ratio with λ-precision.

Lemma 7 *For every $\sigma \geq 1$ and $\lambda > 0$, there exists a constant k with $k = (\frac{2\sigma}{\lambda})^d$ such that $\mathcal{B}_{\sigma,\lambda} \subseteq \mathcal{B}_{k-ply}$.*

Proof. Let B be a ball from a collection of balls with radius in $[1, \sigma]$ and λ-precision, B centered at $\mathbf{x}$. Any ball that could possibly intersect with B must lie in the ball B' of radius 2σ and centered at $\mathbf{x}$. To get an upper bound, we consider the volume of B', which is $c(\sigma)^d$ for some constant c and fixed d. Since the center of any two balls are at least λ-apart, we look another set of balls center at those ball centers with radius $\lambda/2$, it is certain all these new balls are not intersect with each other. Since each new ball of volume $c(\lambda)^d$, the number of new balls could not exceed $(2\sigma/\lambda)^d$. □

Lemma 8 *For any ϵ, there exists a δ such that for any set of balls B, each of radius at least one,*

$$|W_B^\delta| \leq (1+\epsilon)\mu(B)$$

Proof. B. Csikós [5] proved that the volume of the union of some balls in the Euclidean space can not increase if these balls move continuously such that the distances between their centers decrease. If we simply enlarge all centers and radii of all balls in B by $1+\sqrt{d}\delta$ to get $B_{enlarged}$, then move enlarged balls back to their original center to get B_{moved}, we have

$$W_B^\delta \leq W_{B_{moved}}^\delta \leq W_{B_{enlarged}}^\delta \leq (1+\sqrt{d}\delta)^d \mu(B)$$

$$\begin{aligned}(1+\sqrt{d}\delta)^d &= 1 + \sum_{i=1}^{d} \binom{d}{i} (\sqrt{d}\delta)^i \\ &\leq 1 + \sum_{i=1}^{d} \frac{d^i}{2^{i-1}} (\sqrt{d}\delta)^i \\ &\leq 2\frac{1}{1-\frac{d^{3/2}\delta}{2}} - 1.\end{aligned}$$

Let

$$\delta = \frac{2\epsilon}{d\sqrt{d}(2+\epsilon)},$$

then we have

$$W_B^\delta \leq (1+\epsilon)\mu(B).$$

□

Plug in the sphere separator of Miller et al., we get a separator for the Lebesgue measure $\mu(\cdot)$ on the class of ball graphs.

Theorem 9 *For the graph class $\mathcal{B}_{\sigma,\lambda}$ of ball graphs with bounded radius ratio σ and λ-precision. $\forall B \in \mathcal{B}_{\sigma,\lambda}$ and constant α related with d and ϵ only, there exists a sphere separator such that:*

- *$\mu(B_E) \leq \alpha\mu(B)$ where B_E contains the set of all balls in the exterior of S*
- *$\mu(B_I) \leq \alpha\mu(B)$ where B_I contains the set of all balls in the interior of S,*
- *$\mu(B_S) \leq O(\sigma/\lambda, \mu(B)^{1-\frac{1}{d}})$ where B_S contains the set of all balls that intersect S.*

In addition, such an S can be computed by an algorithm in random linear time.

We prove the existence of such a separator by present an algorithm which could find such a separator for a ball graph. For any point set w and ball graph B, we define $B(w)$ as the union of balls intersect any point of w.

geometric separator(B)

- Scale B such that the smallest ball has unit radius.
- Fix any $\epsilon < 1/(d+2)$ and select $\delta = \frac{2\epsilon}{d\sqrt{d}(2+\epsilon)}$ construct the graph H_B^δ according to Lemma 8.
- Run the algorithm of Miller *et al.* on the graph H_B^δ to obtain (H_E, H_S, H_I), hence (W_E, W_S, W_I) where
  1. $|W_S| = O(|W_B^\delta|^{1-1/d})$
  2. $|W_I|, |W_E| \leq \frac{d+1}{d+2}|W_B^\delta|$
- Return the three sets

$$\begin{aligned}B_S &:= B(W_S) \\ B_I &:= B(W_I) \setminus B_S \\ B_E &:= B(W_E) \setminus B_S\end{aligned}$$

Proof.

1. (B_I, B_E, B_S) is a separation of B.
2. The volume of a b of radius r in d-dimension is no more than $6r^d$. For any vertex $w \in W_S$, all balls intersects with w must be inside a ball of radius $2\sigma + \sqrt{d}\delta$.

$$\begin{aligned}\mu(B(w)) &\leq 6(2\sigma+\sqrt{d}\delta)^d \\ &\leq 6(2\sigma+\sqrt{d}\frac{2\epsilon}{d\sqrt{d}(2+\epsilon)})^d \\ &\leq 6(2\sigma+\frac{2\epsilon}{2d})^d \\ &\leq 6(2\sigma+\frac{1}{d^2})^d \\ &\leq 6(2\sigma)^d + 6(2\sigma)^{d-1}\frac{2d}{d^2} \\ &\leq 6(2\sigma)^d + \frac{12(2\sigma)^{d-1}}{d}\end{aligned}$$

If σ, d fixed, let $c = 6(2\sigma)^d + \frac{12(2\sigma)^{d-1}}{d}$ be a constant. We have

$$\begin{aligned}\mu(B_S) &= \mu(\bigcup_{w \in W_S} B(w)) \\ &\leq \sum_{w \in W_S} \mu(B(w)) \\ &\leq c|W_S|\end{aligned}$$

We already have $|W_S| = O(|W_B^\delta|^{1-1/d})$, so $\mu(B_S) = O(|W_B^\delta|^{1-1/d}) = O((\mu(B))^{1-1/d})$.

3.

$$\begin{aligned}\mu(B_E) &\leq |W_E| \\ &\leq \frac{d+1}{d+2}(1+\epsilon)\mu(B) \\ &\leq (1 - \frac{1}{(d+2)^2})\mu(B)\end{aligned}$$

Similarly, we get: $\mu(B_I) \leq (1 - \frac{1}{(d+2)^2})\mu(B)$. □

Note with smaller ϵ, we will have better bound on the split. e.g., $\epsilon < 1/(2d+3)$ will yield $\mu(B_I), \mu(B_E) \leq (1 - \frac{1}{2d+3})\mu(B)$.

This algorithm is of randomized linear time. The running time of this algorithm depends on the running time of the separator theorem of Miller et al., which is randomized linear time.

4 Future work

There are many applications of separators in the counting measure, but not that many in the Lebesgue measure. As the separator theorem is ready, we hope to have more applications use this separator of Lebesgue measure in higher dimensions to solve more problems. More precisely, this algorithm can be used to solve complex problems which can be transformed into a ball graph, which does not have to have a fixed maximum number of overlapped balls. This algorithm requires only ball graphs to be of limited precisions and limited radius ratios.

The more changeling problem would be to drop the bounded radius ratio constraint, or the precision constraint.

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