On Planar Path Transformation

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Abstract

A flip or edge-replacement is considered as a transformation by which one edge e of a geometric object is removed and an edge f ($f \neq e$) is inserted such that the resulting object belongs to the same class as the original object. In this paper, we consider Hamiltonian planar paths as geometric objects. A technique is presented for transforming a given planar path into another one for a set S of n points in convex position in the plane. Under these conditions, we show that any planar path can be transformed into another planar path by at most $2n-5$ flips. For the case when the points are in general position we provide experimental results regarding transformability of any planar path into another. We show that for $n \leq 8$ points in general position any two paths can be transformed into each other. For n points in convex position we show that there are $n2^{n-2}$ directed Hamiltonian planar paths. An algorithm is presented which uses flips of size 1 and flips of size 2 to generate all such paths with $O(n)$ time between the generation of two successive paths.

1 Introduction

The problem of transformations of a certain class of geometric objects consisting of straight line segments and points in the plane by applying small changes or flips (e.g. removing an edge and replacing it with another) in the object has been studied extensively [1], [2], [4], [7]. Given any two objects, two usual questions that are studied are whether the two objects can be transformed to each other and how many transformations are required. In this paper, we study the transformation of Hamiltonian planar paths using flips for a set of points in convex position and in general position in the plane and also determine bounds on the number of transformations needed.

Although flips are studied extensively in triangulations, there are a number of other examples such as spanning trees, Euclidean matchings [5, 6], linked-edge lists, pseudo-triangulations that illustrate this particular computational problem of transformation related to flips. Algorithms for such transformation as well as lower and upper bounds of achieving those transformation results are presented in [2]-[7]. One of the best-known results in the case of planar tree transformation is by Avis and Fukuda [4] who showed that for a point set in general position every planar tree can be transformed into another planar tree by means of at most $2n-4$ flips. Here we study similar problem in planar Hamiltonian path transformation on a set of points in convex position and show that any such path can be transformed into another planar Hamiltonian path by at most $2n-5$ flips. Besides, we show that all planar Hamiltonian paths on a set of at most 8 points in general position can be transformed into other planar paths by single edge flips.

The problem of generating all objects that satisfy a specified property has enjoyed some interest among mathematicians and geometers for many years. There are a number of techniques to handle the problem of enumeration of geometric objects [8]. Also there are some general-purpose strategies for enumeration of combinatorial structures and geometric figures. One such technique is the reverse search scheme due to Avis and Fukuda [4] that has been used to enumerate faces of convex polyhedra, spanning trees of graphs and triangulations of a set of points in the plane, all connected induced subgraphs of a graph etc. A nice treatment of algorithms for enumerating geometric objects can be found in [4, 7]. In [9], Zhu *et. al* an algorithm is described that enumerates simple polygons for a fixed point set which are monotone with respect to the x-direction.

In this paper, we present an algorithm to generate directed planar Hamiltonian paths for points in convex position. We show that for set of n points in convex position there are $n2^{n-2}$ such planar paths. Our algorithm uses edge flips and requires linear space. The time delay between generation of two consecutive paths is also linear.

2 Preliminaries

Let P denote a set of points in the plane in general position. Let $CH(P)$ denote the convex hull of P. Throughout this paper S will denote a set of points in convex position, so all points in S are extreme points of $CH(S)$. The length of a path is defined as the number of edges in the path. Let $\mathcal{P}(P)$ denote the set of all undirected planar Hamiltonian paths on P.

Let $\mathcal{P}_i(P, p)$ denote a set of paths of length i such that

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each path $P' \in \mathcal{P}_i(P, p)$ can be expanded into a path in $\mathcal{P}(P)$ with an endpoint p.

We define the **quality** of a path $\mathcal{P} \in \mathcal{P}(S)$ as the number of edges of $\mathcal{P}$ that lie on $CH(S)$. The quality of a path $\mathcal{P}$ is denoted by $Qua(\mathcal{P})$. A **canonical path** $\mathcal{P}$ is a path in $\mathcal{P}(S)$ with $Qua(\mathcal{P}) = n - 1$.

A **k-flip** is defined as the removal of k edges and the addition of k edges. If $\mathcal{P}$ and $\mathcal{Q}$ are two paths in $\mathcal{P}(P)$ we say that $\mathcal{P}$ can be transformed into $\mathcal{Q}$ by a k-flip if there are edges $e_0, e_1, \ldots, e_{k-1}$ and $f_0, f_1, \ldots, f_{k-1}$ such that $\mathcal{Q} = \mathcal{P} \setminus \{e_0, e_1, \ldots, e_{k-1}\} \cup \{f_0, f_1, \ldots, f_{k-1}\}$. We define the meta graph of P as the graph with vertex set $\mathcal{P}(P)$. There is an edge $(\mathcal{P}, \mathcal{Q})$ in the meta graph if and only if $\mathcal{P}$ can be transformed into $\mathcal{Q}$ by a 1-flip.

3 Transforming Hamiltonian paths

We begin with the observation that if any Hamiltonian path on a convex set of points S has an edge that does not lie on the $CH(S)$, then that edge separates the other points of that path.

Lemma 1 *Let $\mathcal{P} \in \mathcal{P}(S)$ be the path $p_0, p_1, \ldots, p_{n-1}$. The edges (p_0, p_1) and (p_{n-2}, p_{n-1}) lie on CH(S). Moreover, if a path $\mathcal{P}$ has an edge (p_i, p_{i+1}) with $0 < i < n - 2$ that does not lie on CH(S), then the sets $\{p_0, p_1, \ldots, p_{i-1}\}$ and $\{p_{i+2}, p_{i+3}, \ldots, p_{n-1}\}$ lie on different sides of (p_i, p_{i+1}).*

Proof. The result follows from the fact that since $\mathcal{P}$ is planar and S is a convex set of points, no edge can connect two points that lie on different side of (p_i, p_{i+1}). □

The next lemma shows that the quality of any non-canonical path can be increased by one using a 1-flip.

Lemma 2 *A path $\mathcal{P} \in \mathcal{P}(S)$ with $Qua(\mathcal{P}) < n - 1$ can be transformed by a 1-flip into a path $\mathcal{Q}$ with $Qua(\mathcal{Q}) = Qua(\mathcal{P}) + 1$.*

Proof. Let $\mathcal{P}$ be the path $p_0, p_1, \ldots, p_{n-1}$. Let i be the smallest integer such that $i > 1$ and p_i is a neighbour of p_0 on $CH(S)$. If $i = n-1$ then from Lemma 1 we know that all edges of $\mathcal{P}$ lie on $CH(S)$, i.e., $Qua(\mathcal{P}) = n - 1$. So $i < n - 1$. Since $i > 1$ the edge (p_0, p_i) is not in $\mathcal{P}$. If the edge (p_{i-1}, p_i) lies on $CH(S)$, then the path from p_0 to p_i contains an edge e not on $CH(S)$. Vertices p_0 and p_i lie on the same side of e, contradicting Lemma 1. Therefore (p_{i-1}, p_i) does not lie on $CH(S)$. Construct $\mathcal{Q}$ by removing edge (p_{i-1}, p_i) from and add edge (p_0, p_i) to $\mathcal{P}$. So $\mathcal{Q} = p_{i-1}, p_{i-2}, \ldots, p_1, p_0, p_i, p_{i+1}, \ldots, p_{n-1}$. Since (p_0, p_i) lies on $CH(S)$ and (p_{i-1}, p_i) does not, we have $Qua(\mathcal{Q}) = Qua(\mathcal{P}) + 1$. □

Lemma 3 *A path $\mathcal{P} \in \mathcal{P}(S)$ can be transformed into a path $\mathcal{Q} \in \mathcal{P}(S)$ using at most $2n - 5$ 1-flips.*

Proof. From Lemma 1 we see that $Qua(\mathcal{P}) \geq 2$ and $Qua(\mathcal{Q}) \geq 2$. So Lemma 2 shows that we can transform both $\mathcal{P}$ and $\mathcal{Q}$ into canonical paths in at most $n - 3$ steps each. Since any canonical path can be transformed into another canonical path by a single flip, the result follows. □

3.1 Experimental Results: Transforming Hamiltonian paths on points in general position

In this section, we present experimental results that will show that two planar paths on any set of $n (n \leq 8)$ points in general position (with no three points collinear) can be transformed into each other using 1-flips.

We base our analysis on the enumeration of all combinatorially inequivalent sets of points as described in [10]. The order type of a set $\{p_0, p_1 \cdots, p_{n-1}\}$ of n points in general position is a mapping that assigns to each ordered triple i, j, k in $\{0, 1, \cdots, n-1\}$ the orientation (either clockwise or counterclockwise) of the triple p_i, p_j, p_k. Two point sets P_0 and P_1 with n points each are said to be combinatorially equivalent if the points in P_0 as well as the points in P_1 can be labeled with the same set of n labels so that they have the same order types [10],[11].

Lemma 4 *Two point sets P_0 and P_1 with the same order types have the same meta graphs.*

Proof. Assume the points in both P_0 and P_1 are labeled with labels $\{p_0, p_1, \cdots, p_{n-1}\}$ such that P_0 and P_1 have the same order type. Consider a planar path $\mathcal{P}$ in P_0 and the same path $\mathcal{Q}$ in P_1. Suppose that in $\mathcal{P}$ the edges (p_h, p_i) and (p_j, p_k) intersect. So in P_0 points p_j and p_k lie on opposite sides of the line through p_h and p_i. Therefore (p_h, p_j, p_i) and (p_h, p_k, p_i) have opposite orientations. Similarly p_h and p_i lie on opposite sides of the line through p_j and p_k, so (p_j, p_h, p_k) and (p_j, p_i, p_k) have opposite orientations. Since P_0 and P_1 have the same order type, this implies that (p_h, p_i) and (p_j, p_k) intersect in P_1. Therefore edges in P_0 intersect each other if and only if they intersect each other in P_1. So any path that is planar in P_0 is also planar in P_1 and the meta graphs of P_0 and P_1 have the same vertex set.

If two planar paths in P_0 can be transformed into each other by a 1-flip, then the same two paths in P_1 can be transformed into each other by the same 1-flip. So the meta graphs of P_0 and P_1 have the same edge set.

□

For each combinatorially distinct set of n points with $n \leq 8$ given in [11] we generate all Hamiltonian paths consisting of n points and compute the set of permutations that correspond to the planar paths. These paths forms the vertex set of the meta graph. We find the edge set of the meta graph by determining which two planar paths can be transformed into each other by a

1-flip. Finally we determine whether the meta graph is connected. For an illustration, see Figure 1 which shows the meta graph of paths of a set of 4 points.

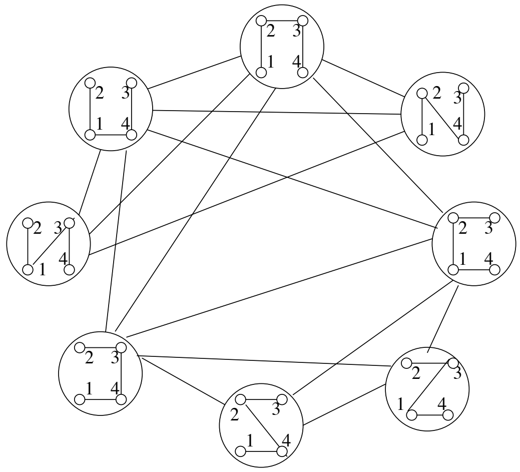

Figure 1: The meta graph of a set of 4 points.

Based on these experiments we prove the following lemma.

Lemma 5 *The meta graph for $n \leq 8$ points in general position is connected.*

We also form the meta graphs for larger sets of points and in all cases found connected meta graphs. We dare to conjecture that the meta graph for any set of points, even those not in general position, is connected.

4 Counting Hamiltonian paths on convex point sets

In this section we show that the size of $\mathcal{P}(S)$ is $n2^{n-3}$. We start with counting the number of paths $\mathcal{P}_i(S,p)$. Let p_0 be a point of S. Let c_i be the number of paths in $\mathcal{P}_i(S,p_0)$. Clearly the value of c_i is independent of p_0. The extremal points in a subset of neighbouring points in S are called the first and last points of the subset. The names refer to the order in which we encounter these points if we traverse S counter clockwise.

Lemma 6 *We have of $c_i = 2^i$ for $0 \leq i < n-1$ and $c_{n-1} = 2^{n-2}$.*

Proof. We prove this lemma by induction. Clearly the lemma holds for $i = 0$. Assume the lemma holds for $0 \leq i < k$ with $0 < k \leq n-2$. We now show that the lemma holds for $i = k$. Consider a path $\mathcal{P} = p_0, p_1, \ldots p_i \in \mathcal{P}_i(S,p_0)$. Recall that $\mathcal{P}$ is a subgraph of a planar Hamiltonian path with p_0 as one of its end points. From Lemma 1 we derive that $\{p_0, p_1, \ldots p_i\}$ are neighbours on $CH(S)$. Moreover, the point p_i has to be the first or the last point of $\{p_0, p_1, \ldots p_i\}$.

For the same reason, p_{i-1} is the first or the last point of $\{p_0, p_1, \ldots p_{i-1}\}$. So for each path in $\mathcal{P}_{i-1}(S,p_0)$ we can have two paths in $\mathcal{P}_i(S,p_0)$: one in which p_{i-1} and p_i are neighbours on $CH(S)$ and one in which they are not. This implies that $c_i = 2 * c_{i-1}$. Since $c_0 = 1$ we have $c_i = 2^i$ for $0 < i < n-1$. The final part of the lemma holds since $c_{n-1} = c_{n-2} = 2^{n-2}$. □

Now we prove the main theorem:

Theorem 7 *The number of undirected Hamiltonian paths on a set of points in convex position is $n2^{n-3}$.*

Proof. From Lemma 6 we know that the number of Hamiltonian paths of S with an end point p_0 is equal to 2^{n-2}. Since there n different end points and each path has two end points, the proof follows.

□

5 Directed Path Construction Algorithm

Let $D(S)$ be the set of all directed planar Hamiltonian paths on S. We present an algorithm for the generation of $D(S)$. A directed path $A \in D(S)$ can be associated with an undirected path $A' \in \mathcal{P}(S)$. Let us define a **k-flip** in directed paths. If $A, B \in D(S)$ and $A', B' \in \mathcal{P}(S)$ we say that A is transformed into B by a $k-$flip if there is a $k-$flip that transforms A' into B'.

Consider a path $D = (p_0, p_1, \cdots, p_{n-1}) \in D(S)$. We define an encoding $[p_0, b_1, b_2, \ldots, b_{n-1}]$ of D as follows: The first element is the first point of the path. The value of b_i for $1 \leq i < n-1$ is 0 or 1 if p_i is the first or last point respectively of the points $\{p_i, p_{i+1}, \ldots, p_{n-1}\}$. Since an encoding of a path in $D(S)$ contains one point and $n-2$ bits, we have $|D(S)| = n2^{n-2}$, which also follows from Theorem 7 which deals with undirected paths.

Now we show that beginning from a canonical path with encoding $[p_0, 0, 0, 0, \cdots]$ we can generate all paths that start at p_0 using 1-flips and 2-flips. The last path generated has encoding $[p_0, 1, 1, 1, \cdots]$. After that we can generate a path with encoding $[p_i, 0, 0, 0, \cdots]$, where p_i is any point different from p_0.

Let $[p_0, b_1, b_2, \ldots, b_{n-2}]$ denote the most recently generated path. The following three cases define the next path in the generation:

Case one: $b_{n-2} = 0$**.** The next path is $[p_0, b_1, b_2, \ldots, b_{n-3}, b'_{n-2}]$ with $b'_{n-2} = 1$.

Case two: $b_k = 0$ **and** $b_t = 1$ **for** $k+1 \leq t < n-1$**.** The next path is $[p_0, b_1, b_2, \ldots, b_{k-1}, b'_k, b'_{k+1}, \ldots, b'_{n-2}]$ with $b'_t = 1 - b_t$ for $k \leq t < n-1$.

Case three: $b_t = 1$ **for** $1 \leq t < n-1$**.** The next path is $[p_i, 0, 0, 0, \ldots]$, where p_i is any point in S.

Lemma 8 *The paths generated in case one, two and three can be achieved with a 1-flip, a 2-flip and a 1-flip respectively.*

Figure 2 shows an example of the first two cases.

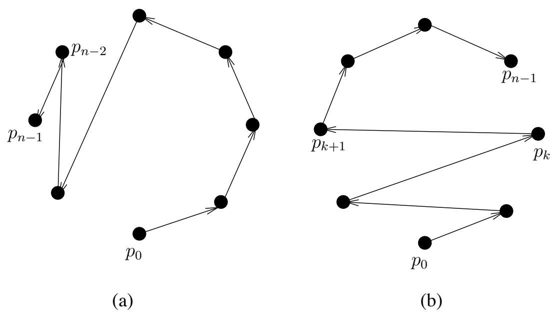

Figure 2: (a) Showing an example of case one with encoding $[p_0, b_1, b_2, \ldots, 0]$ (b) and case two with encoding $[p_0, b_1, b_2, \ldots, b_{k-1}, b_k, b_{k+1}, \ldots, b_{n-2}]$ with $b_k = 0$ and $b_t = 1$, for $k+1 \leq t < n-1$.

5.1 Generation Method

Let $S = \{p_0, p_1, \ldots, p_{n-1}\}$. We start generating the sequence of planar Hamiltonian directed paths with the path $[p_0, 0, 0, 0, \cdots]$. As shown above we can generate all 2^{n-2} directed paths which begin at p_0. After that we can generate the path with encoding $[p_1, 0, 0, 0, \cdots]$. This process is continued until we have reached path $[p_{n-1}, 1, 1, 1, \cdots]$. Since given a path we can generate the next path, it is clear that the algorithm requires $O(n)$ space. Also the time between the generation of successive paths is $O(n)$.

6 Conclusion and Open Problem

The problem of transforming planar Hamiltonian paths for points in general position and points in convex position is considered. In the case where the points are in convex position it is proved that at most $(2n-5)$ edge changes are sufficient for any two planar paths to be transformed from one to another. The main implication of this result is that the meta graph of $\mathcal{P}(S)$ is connected and the diameter of the meta graph is bounded by $2n-5$. In addition, experimental results for the case of a small number of points in general position indicate that, for all sets with at most 8 points, the corresponding meta graphs are connected. A recursive technique is used to show that the number of directed Hamiltonian planar paths when the points are in convex position equals $n2^{n-2}$. An algorithm is also presented that allows us to generate uniquely the set of all directed paths of a set of points in convex position employing flips of size one and flips of size two. Finally we conjecture that the meta graph is always connected for a set of points in general position. Settling this conjecture remains an interesting and challenging open problem.

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