Note on the pair-crossing number and the odd-crossing number

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Abstract

The crossing number $\text{CR}(G)$ of a graph G is the minimum possible number of edge-crossings in a drawing of G, the pair-crossing number $\text{PCR}(G)$ is the minimum possible number of crossing pairs of edges in a drawing of G, and the odd-crossing number $\text{OCR}(G)$ is the minimum number of pairs of edges that cross an odd number of times. Clearly, $\text{OCR}(G) \leq \text{PCR}(G) \leq \text{CR}(G)$. We construct graphs with $0.855 \cdot \text{PCR}(G) \geq \text{OCR}(G)$. This improves the bound of Pelsmajer, Schaefer and Štefankovič. Our construction also answers an old question of Tutte.

Slightly improving the bound of Valtr, we also show that if the pair-crossing number of G is k, then its crossing number is at most $O(k^2/\log^2 k)$.

1 Introduction

In a *drawing* of a graph G vertices are represented by points and edges are represented by Jordan curves connecting the corresponding points. If it does not lead to confusion, we do not make any notational distinction between vertices (resp. edges) and points (resp. curves) representing them. We assume that the edges do not pass through vertices, any two edges have finitely many common points and each of them is either a common endpoint, or a proper crossing. We also assume that no three edges cross at the same point.

The *crossing number* $\text{CR}(G)$ is the minimum number of edge-crossings (i. e. crossing points) over all drawings of G. The *pair-crossing number* $\text{PCR}(G)$ is the minimum number of crossing pairs of edges over all drawings of G, and the *odd-crossing number* $\text{OCR}(G)$ is the minimum number of pairs of edges that cross an odd number of times over all drawings of G.

Clearly, for any graph G we have

$$\text{OCR}(G) \leq \text{PCR}(G) \leq \text{CR}(G).$$

Pach and Tóth [PT00a] proved that $\text{CR}(G)$ cannot be arbitrarily large if $\text{OCR}(G)$ is bounded, namely, for any G, if $\text{OCR}(G) = k$, then $\text{CR}(G) \leq 2k^2$ and this is the best known bound. Obviously it follows that $\text{PCR}(G) \leq 2k^2$ as well and this is also the best known bound. On the other hand, Pelsmajer, Schaefer and Štefankovič [PSS06] proved that $\text{OCR}(G)$ and $\text{PCR}(G)$ are not necessarily equal, they constructed a series of graphs with

$$\text{OCR}(G) < \left(\frac{\sqrt{3}}{2} + o(1)\right) \cdot \text{PCR}(G).$$

We slightly improve their bound with a completely different construction.

Theorem 1. *There is a series of graphs G with*

$$\text{OCR}(G) < \left(\frac{3\sqrt{5}}{2} - \frac{5}{2} + o(1)\right) \cdot \text{PCR}(G).$$

Note that $\frac{\sqrt{3}}{2} \approx 0.866$ and $\frac{3\sqrt{5}}{2} - \frac{5}{2} \approx 0.855$. There are many other versions of the crossing number (see e. g. [PT00b], [PSS05]). Tutte [T70] defined the following version which we call *independent algebraic crossing number*, $\text{IACR}(G)$, and we also define its close relative the *algebraic crossing number*, $\text{ACR}(G)$.

Orient the edges of G arbitrarily. For any drawing D of G, and any two edges e and f, let c^+ (resp. c^-) be the number of e–f crossings where e crosses f from left to right (resp. from right to left). Let $c(e,f) = |c^+ - c^-|$, and let $c(D) = \sum c(e,f)$ where the summation is for all pairs of *independent* edges. Similarly, let $c'(D) = \sum c(e,f)$ where the summation is for *all* pairs of edges. Finally, let $\text{IACR}(G)$ be the minimum of $c(D)$ for all drawings D of G, and let $\text{ACR}(G)$ be the minimum of $c'(D)$ for all drawings D of G.

It is easy to see that for any graph G we have $\text{IACR}(G) \leq \text{ACR}(G)$ and

$$\text{OCR}(G) \leq \text{ACR}(G) \leq \text{CR}(G).$$

In the construction of Pelsmajer, Schaefer and Štefankovič [PSS06] for each of the graphs the pair-crossing number and the algebraic crossing number are equal. Therefore, for their series of graphs

$$\text{OCR}(G) < \left(\frac{\sqrt{3}}{2} + o(1)\right) \cdot \text{ACR}(G).$$

We show that $\text{ACR}(G)$ and $\text{PCR}(G)$ are not necessarity equal either.

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Theorem 2. *There is a series of graphs G with*

$$\textsc{acr}(G) < \left(\frac{3\sqrt{5}}{2} - \frac{5}{2} + o(1)\right) \cdot \textsc{pcr}(G).$$

Since $\textsc{ocr}(G) \leq \textsc{acr}(G)$ for every graph G, Theorem 1 is an immediate consequence of Theorem 2. Tutte [T70] asked if $\textsc{iacr}(G) = \textsc{cr}(G)$ holds for every graph G. Since $\textsc{iacr}(G) \leq \textsc{acr}(G)$, Theorem 2 gives a negative answer for this question. Finally, since $\textsc{pcr}(G) \leq \textsc{cr}(G)$, Theorems 1 and 2 hold also for $\textsc{cr}(G)$ instead of $\textsc{pcr}(G)$. Moreover, the whole argument works, without any change.

It is still a challenging open question whether $\textsc{cr}(G) = \textsc{pcr}(G)$ holds for all graphs G. Pach and Tóth [PT00a] proved that for any G, if $\textsc{pcr}(G) = k$, then $\textsc{cr}(G) \leq 2k^2$. Valtr [V05] managed to improve this bound to $\textsc{cr}(G) \leq 2k^2/\log k$. Based on the ideas of Valtr, in this note we give a further little improvement.

Theorem 3. *For any graph G, if $\textsc{pcr}(G) = k$, then $\textsc{cr}(G) \leq 9k^2/\log^2 k$.*

The proof of Theorem 3 is omitted from this version, but it is available in the electronic version.

2 Proof of Theorem 2

The idea and sketch of the construction. For simplicity, we write *alg-crossing number* for the algebraic crossing number. In the descriprion we use weights on the edges of the graph. If we substitute each weighted edge by an appropriate number of parallel paths, say, each of length two, we can obtain an unweighted simple graph whose ratio of the pair-crossing and alg-crossing numbers is arbitrarily close to that of the weighted construction.

First of all, take a "frame" F, which is a cycle K with very heavy edges, together with a vertex V connected to all vertices of the cycle, also with very heavy edges. In the optimal drawings the edges of F do not participate in any crossing, and we can assume that V is drawn *outside* the cycle K. Therefore, all additional edges and vertices of the graph will be *inside* K.

We have four further vertices, each connected to three different vertices of the frame-cycle K. These three edges have weights 1, 1, w respectively, with some $1 < w < 2$. Each one of these four vertices, together with the adjacent three edges, and the frame F, is called a *component* of the construction.

If we take any *two* of the components, it is easy to see how to draw them optimally, both in the alg-crossing and pair-crossing sense. See Figure 1. The point is that if we take all four components, we can still draw them such that each of the six pairs are drawn optimally, in the alg-crossing sense. See Figure 2. On the other hand, it is easy to see that it is impossible to draw all six pairs optimally in the pair-crossing sense, some pairs will not have their best drawing. See Figure 3. Note that we did not indicate vertex V of the frame.

We get the best result with $w = \frac{\sqrt{5}+1}{2}$. Actually, we will see that among any *three* components there is a pair which is not drawn optimally in the pair-crossing sense. So, we could take the union of just three components, but that gives a weaker bound.

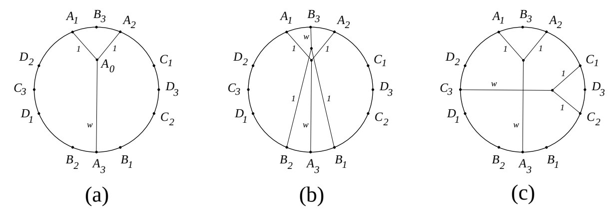

Figure 1: (a) Component A (b), (c) Optimal drawings of the pairs (A, B) and (A, C), resp.

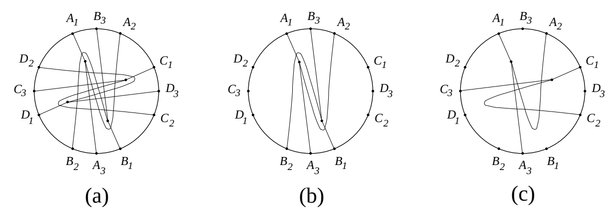

Figure 2: (a) Optimal drawing of G in the alg-crossing sense (b), (c) The pairs (A, B) and (A, C) resp. from the same drawing.

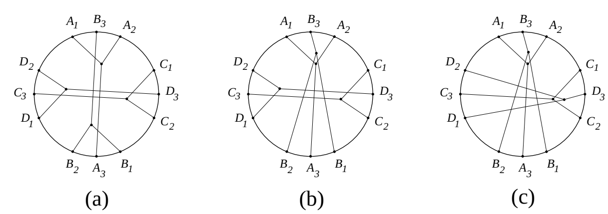

Figure 3: (a), (b) Cases 1 and 2 of Lemma 2, resp., optimal drawings of G in the pair-crossing sense (c) Case 3, not optimal drawing.

Proof of Theorem 2.

A *weighted graph* G is a graph with positive weights on its edges. For any edge e let $w(e)$ denote its weight. For any fixed drawing $\mathcal{G}$ of G, the *pair-crossing value* $\textsc{pcr}(\mathcal{G}) = \sum w(e)w(f)$ where the sum goes over all crossing pairs of edges e, f. For the *alg-crossing value* $\textsc{acr}(\mathcal{G})$, orient the edges of G arbitrarily, let c^+ (resp.

c^-) be the number of e–f crossings where e crosses f from left to right (resp. from right to left), let $c(e,f) = |c^+ - c^-|$. The *alg-crossing value* $\text{ACR}(\mathcal{G}) = \sum w(e)w(f)c(e,f)$ where the sum goes over all pairs of edges e, f.

The pair-crossing number (resp. alg-crossing number) is the minimum of the pair-crossing value (resp. alg-crossing value) over all drawings. That is,

$$\text{PCR}(G) = \min_{\text{all drawings}} \sum_{\substack{\text{all crossing pairs}\\ \text{of edges } e,f}} w(e)w(f),$$

$$\text{ACR}(G) = \min_{\text{all drawings}} \sum_{\substack{\text{all pairs}\\ \text{of edges } e,f}} w(e)w(f)c(e,f).$$

Theorem 4. *There exists a weighted graph G with* $\text{PCR}(G) = (\frac{3\sqrt{5}}{2} - \frac{5}{2}) \cdot \text{ACR}(G)$.

Proof of Theorem 4. First we define the weighted graph G. Take nine vertices, A_1, B_3, A_2, C_1, D_3, C_2, B_1, A_3, B_2, D_1, C_3, D_2 which form cycle K in this order. Vertex V is connected to all of the nine vertices of K. These vertices and edges form the "frame" F. All edges of F have extremely large weights, therefore, they do not participate in any crossing in an optimal drawing. We can assume without loss of generality that V is drawn *outside* the cycle K, so all further edges and vertices of G will be *inside* K.

There are four more vertices, A_0, B_0, C_0, D_0, and for $X = A, B, C, D$, X_0 is connected to X_1, X_2, and X_3. The weight $w(X_0X_1) = w(X_0X_2) = 1$ and $w(X_0X_3) = w = \frac{\sqrt{5}+1}{2}$. Graph X is a subgraph of G, induced by the frame and X_0. See Figure 1. Finally, for any $X, Y = A, B, C, D$, $X \neq Y$, let $\text{PCR}(X,Y) = \text{PCR}(X \cup Y)$, and $\text{ACR}(X,Y) = \text{ACR}(X \cup Y)$.

First we find all these crossing numbers. Moreover, we also find the second smallest pair-crossing values.

Start with $A \cup C$. Since the path $A_1B_3A_2$ is not intersected by any edge in an optimal drawing, we can contract it to one vertex, without changing the pair-crossing number, so now $A_1 = A_2$. Consider the edges $e_1 = A_1A_0$ and $e_2 = A_2A_0$. Now they connect the same vertices. Suppose that they do not go parallel in an optimal drawing. Let $w^*(e_1)$ (resp. $v^*(e_2)$) be the sum of the weights of the edges crossing e_1 (resp. e_2) and assume without loss of generality that $w^*(e_1) \leq w^*(e_2)$. Then draw e_2 parallel with e_1, the drawing obtained is at least as good as the original drawing was, so it is optimal as well. Therefore, we can assume without loss of generality that e_1 and e_2 go parallel in an optimal drawing, so we can substitute them by one edge of weight 2. Similarly, we can contract the path $C_1D_3C_2$ and substitute the edges C_1C_0 and C_2C_0 by one edge of weight 2. Now we have a very simple graph, whose pair-crossing number is immediate, that is, we have two paths $C_1C_0C_3$ and $A_1A_0A_3$, which have to cross each other, and on both paths one edge has weight w, the other one has weight 2. Since $w < 2$, in the optimal drawing the edges A_0A_3 and C_0C_3 will cross each other and no other edges cross so we have $\text{PCR}(A,C) = w^2$. Moreover, it is also clear that the second smallest pair-crossing value is $2w$.

The same argument holds for $\text{ACR}(A,C)$, moreover, by symmetry, we can argue exactly the same way for the pairs (A,D), (B,C), and (B,D).

Now we determine $\text{PCR}(A,B)$ and the second smallest pair-crossing value. The edges $a_1 = A_0A_1$, $a_2 = A_0A_2$, $a_3 = A_0A_3$ divide the interior of F into three regions R_1, R_2 and R_3. Number them in such a way that for $i = 1,2,3$, a_i is outside R_i. See Fig. 1. Once we place B_0 into one of these regions, it is clear how to draw the edges $b_1 = B_0BB_1$, $b_2 = B_0BB_2$, $b_3 = B_0BB_3$ to get the best of the possible drawings. If B_0 is in R_1 or in R_2, we get the pair-crossing value $2w$, but if we place B_0 in R_3, then we get 2. Again, the same argument holds for $\text{ACR}(A,B)$, and by symmetry, the situation is the same with the pair (C,D). See Figure 1.

Lemma 1.
$$\text{ACR}(G) = 4w^2 + 4.$$

Proof of Lemma 1. We have $\text{ACR}(G) \geq \text{ACR}(A,B) + \text{ACR}(A,C) + \text{ACR}(A,D) + \text{ACR}(B,C) + \text{ACR}(B,D) + \text{ACR}(C,D) = 4w^2 + 4$, and there is a drawing (see Fig. 2) with exactly this alg-crossing value. $\square$

Lemma 2.
$$\text{PCR}(G) = 4w^2 + 4w.$$

Proof of Lemma 2. The argument, except for the exact calculation, should be clear from the figures. While we have a drawing which is optimal for all six pairs in the alg-crossing sense (see Fig. 2), in the pair-crossing sense some of the pairs will not be optimal, they have to take at least the second smallest pair-crossing value. We start with an observation that in any *triple* at least one pair is not optimal. Then we will distinguish three cases.

Suppose we have a drawing $\mathcal{G}$ of G where pairs (A,C) and (A,D) are drawn optimally, that is, $\text{PCR}(\mathcal{A},\mathcal{C}) = \text{PCR}(\mathcal{A},\mathcal{D}) = w^2$. Recall that the edges $a_1 = A_0A_1$, $a_2 = A_0A_2$, $a_3 = A_0A_3$ divide the interior of F into three regions R_1, R_2 and R_3. It follows from the above argument that $C_0 \in R_1$, $D_0 \in R_2$. But then the pair (C,D) is not drawn optimally, that is, $\text{PCR}(\mathcal{C},\mathcal{D}) > 2$, so we have $\text{PCR}(\mathcal{C},\mathcal{D}) \geq 2w$. In other words, it is impossible that all three pairs (A,C), (A,D), (C,D) are drawn optimally at the same time. By symmetry, this observation holds for any triple of A, B, C, D.

We have to distinguish three cases.

Case 1. Neither (A,B), nor (C,D) are drawn optimally. In this case, $\text{PCR}(\mathcal{A},\mathcal{B}) > 2$ so by the previous argument we have $\textsc{pcr}(\mathcal{A},\mathcal{B}) \geq 2w$, and similarly $\textsc{pcr}(\mathcal{C},\mathcal{D}) \geq 2w$. For all other pairs we have pair-crossing value at least w^2, therefore, $\textsc{pcr}(\mathcal{G}) = \textsc{pcr}(\mathcal{A},\mathcal{B}) + \textsc{pcr}(\mathcal{A},\mathcal{C}) + \textsc{pcr}(\mathcal{A},\mathcal{D}) + \textsc{pcr}(\mathcal{B},\mathcal{C}) + \textsc{pcr}(\mathcal{B},\mathcal{D}) + \textsc{pcr}(\mathcal{C},\mathcal{D}) \geq 4w^2 + 4w$.

Case 2. (A, B) is drawn optimally, (C, D) is not. Since (A, B) is drawn optimally, one of the pairs (A, C) and (B, C) and one of the pairs (A, D) and (B, D) is not drawn optimally so we have $\textsc{pcr}(\mathcal{A},\mathcal{C}) + \textsc{pcr}(\mathcal{B},\mathcal{C}) \geq w^2 + 2w$ and analogously $\textsc{pcr}(\mathcal{A},\mathcal{D}) + \textsc{pcr}(\mathcal{B},\mathcal{D}) \geq w^2 + 2w$, therefore, $\textsc{pcr}(\mathcal{G}) = \textsc{pcr}(\mathcal{A},\mathcal{B}) + \textsc{pcr}(\mathcal{A},\mathcal{C}) + \textsc{pcr}(\mathcal{A},\mathcal{D}) + \textsc{pcr}(\mathcal{B},\mathcal{C}) + \textsc{pcr}(\mathcal{B},\mathcal{D}) + \textsc{pcr}(\mathcal{C},\mathcal{D}) \geq 2w^2 + 6w + 2 = 4w^2 + 4w$. The last equality can be verified by solving the quadratic equation.

Case 3. Both (A, B) and (C, D) are drawn optimally. If none of the other four pairs is optimal, then we have $\textsc{pcr}(\mathcal{G}) = \textsc{pcr}(\mathcal{A},\mathcal{B}) + \textsc{pcr}(\mathcal{A},\mathcal{C}) + \textsc{pcr}(\mathcal{A},\mathcal{D}) + \textsc{pcr}(\mathcal{B},\mathcal{C}) + \textsc{pcr}(\mathcal{B},\mathcal{D}) + \textsc{pcr}(\mathcal{C},\mathcal{D}) \geq 8w+4 = 4w^2+4w$. So we can assume that one of them, say (A, C) is drawn optimally, that is, $\textsc{pcr}(\mathcal{A},\mathcal{C}) = w^2$. Since in any triple we have at least one non-optimal pair, we have $\textsc{pcr}(\mathcal{B},\mathcal{C}) \geq 2w$ and $\textsc{pcr}(\mathcal{A},\mathcal{D}) \geq 2w$. We estimate $\textsc{pcr}(\mathcal{B},\mathcal{D})$ now.

Again, the edges $a_1 = A_0A_1$, $a_2 = A_0A_2$, $a_3 = A_0A_3$ of A divide the interior of F into three regions R_1, R_2 and R_3 with R_i is the one to the opposite of a_i. Similarly define the regions Q_1, Q_2, Q_3 for C. Since (A, C) is drawn optimally, R_3 and Q_3 are disjoint. Since (A, B) is drawn optimally, $B_0 \in R_3$, and since (C, D) is also drawn optimally, $D_0 \in Q_3$. See Figure 3. Now it is not hard to see that the edge D_0D_1 either crosses A_0A_1, A_0A_2, and B_0B_3, or B_0B_1, B_0B_2, and A_0A_3. The same holds for the edge D_0D_1, so $\textsc{pcr}(\mathcal{A},\mathcal{D}) + \textsc{pcr}(\mathcal{B},\mathcal{D}) \geq 2w + 4$. So we have $\textsc{pcr}(\mathcal{G}) = \textsc{pcr}(\mathcal{A},\mathcal{B}) + \textsc{pcr}(\mathcal{A},\mathcal{C}) + \textsc{pcr}(\mathcal{A},\mathcal{D}) + \textsc{pcr}(\mathcal{B},\mathcal{C}) + \textsc{pcr}(\mathcal{B},\mathcal{D}) + \textsc{pcr}(\mathcal{C},\mathcal{D}) \geq w^2 + 4w + 8 > 4w^2 + 4w$. This concludes the proof of Lemma 2. □.

Now we have

$$\frac{\textsc{acr}(G)}{\textsc{pcr}(G)} = \frac{4w^2 + 4}{4w^2 + 4w} = \frac{-5}{2} + \frac{3\sqrt{5}}{2},$$

and Theorem 4 follows immediately. □

Proof of Theorem 2. Let $\varepsilon > 0$ an arbitrary small number. Let p and q be positive integers with the property that $w(1 + \frac{\varepsilon}{10}) > \frac{p}{q} > w(1 - \frac{\varepsilon}{10})$. Let G_ε be the following graph. In the weighted graph G of Theorem 4, (i) substitute each edge $e = XY$ of weight 1 with q paths between X and Y, each of length 2, (ii) substitute each edge $e = XY$ of weight w with p paths between X and Y, each of length 2, and (iii) substitute each edge $e = XY$ of the frame F with a huge number of paths between X and Y, each of length 2. Then

$$\frac{\textsc{acr}(G_\varepsilon)}{\textsc{pcr}(G_\varepsilon)} < \frac{\textsc{acr}(G)}{\textsc{pcr}(G)}(1 + \varepsilon) < \frac{-5}{2} + \frac{3\sqrt{5}}{2} + \varepsilon. \qquad \square$$

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References

[PT00a] J. Pach, G. Tóth, Which crossing number is it anyway? *J. Combin. Theory Ser. B* **80**, (2000), 225-246.

[PT00b] J. Pach, G. Tóth: Thirteen problems on crossing numbers, *Geombinatorics* **9** (2000), 194-207.

[PSS06] M. Pelsmajer, M. Schaefer, D. Štefankovič, Odd crossing number is not crossing number, In: Graph Drawing 2005 (P. Healy, N. S. Nikolov, eds.), Lecture Notes in Computer Science **3843**, Springer-Verlag, Berlin, 2006, 386-396.

[PSS05] M. Pelsmajer, M. Schaefer, D. Štefankovič, Removing even crossings, in: European Conference on Combinatorics, Graph Theory and Applications (EuroComb '05), (S. Felsner ed.) DMTCS Conference Volume AE (2005), 105-110.

[SS04] M. Schaefer, D. Štefankovič, Decidability of string graphs, *J. Comput. System Sci.* **68** (2004), 319-334.

[T70] W. T. Tutte, Toward a theory of crossing numbers, *J. Combin. Theory* **8**, (1970), 45-53.

[V05] P. Valtr, On the pair-crossing number, In: Combinatorial and computational geometry, Math. Sci. Res. Inst. Publ., **52**, 569-575, Cambridge Univ. Press, Cambridge, 2005.