On the Fewest Nets Problem for Convex Polyhedra

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Abstract

Given a convex polyhedron with n vertices and F faces, what is the fewest number of pieces, each of which unfolds to a simple polygon, into which it may be cut by slices along edges? Shephard's conjecture says that this number is always 1, but it's still open. The fewest nets problem asks to provide upper bounds for the number of pieces in terms of n and/or F. We improve the previous best known bound of $F/2$ by proving that every convex polyhedron can be unfolded into no more than $3F/8$ non-overlapping nets. If the polyhedron is triangulated, the upper bound we obtain is $4F/11$.

1 Introduction

The classic problem whether a convex polyhedron can be cut along edges and unfolded on a plane into a non-overlapping net is still open. While this classic problem goes back to Albrecht Dürer [6] in the 16th century, the first formal description of the problem is due to Shephard [11], who in 1975 stated the following:

Conjecture 1 *Every convex polyhedron can be cut along edges and unfolded into a non-overlapping net.*

The above statement is false for non-convex polyhedra. An example of an ununfoldable polyhedron can be found in [2].

If the constraint of unfolding the polyhedron into a single connected piece is loosened, we obtain the fewest nets problem, which was proposed by Joseph O'Rourke [4] at the Canadian Conference in Computational Geometry CCCG 2003:

The Fewest Nets Problem Given a convex polyhedron of n vertices and F faces, what is the fewest number of pieces, each of which unfolds to a simple polygon, into which it may be cut by slices along edges? Provide an upper bound as a function of n and/or F.

Shephard's Conjecture states that this number is always 1, but it is still open. The trivial upper bound F can be obtained by cutting out each face. If the polyhedron is *simplicial* (every face is a triangle), the dual graph of the polyhedron is a cubic bridgeless graph, therefore by Peterson's theorem, it has a perfect matching. Then pairs of matched triangular faces will unfold without overlap, which leads to $F/2$ pieces.

If the polyhedron is *simple* (every vertex has degree 3), the dual graph is planar and triangulated, therefore by a result due to Biedl et al. [3], it has a matching of size $(F + 4)/3$. Then pairs of matched faces will unfold without overlap, and the number of unmatched faces will be at most $(F - 8)/3$. This leads to at most $(F + 4)/3 + (F - 8)/3 = 2/3(F - 2)$ pieces.

The first bound for general polyhedra was obtained by Michael Spriggs (personal communication to E. Demaine and J. O'Rourke - see Chapter 22 of [5]), who established in 2003 that all convex polyhedra can be unfolded into $2F/3$ non-overlapping nets. His argument depends on a result by D. Barnette [1] that every 3-connected planar graph has a spanning tree of maximum degree 3. Then by using this result for the dual graph of the polyhedron (which is 3-connected and planar according to Steinitz's Theorem), one can match some of the faces by plucking them off from the leaves of the spanning tree, and obtain a bound of $2F/3$. The smallest bound obtained so far is $F/2$, and it was obtained by Vida Dujmovic, Pat Morin and David Wood in 2004.(personal communication to E. Demaine and J. O'Rourke - see Chapter 22 of [5]) In this paper we use graph domination to show that every convex polyhedron can be unfolded into no more that $3F/8$ non-overlapping nets. For simplicial polyhedra, the upper bound we obtain is $4F/11$. In Section 2 we generalize the volcano unfolding of domes from [5]. The graph theoretic results on graph domination that we are using in our proofs are presented in Section 3, and the main results are presented in Section 4.

2 Volcano Unfoldings

We define a *polyhedron* to be a connected set of closed planar polygons called *faces* in the 3-dimensional space such that: (1) the intersection of two faces is a set of vertices and edges common to both polygons (2) every edge is shared by at most two faces. All polygons in this paper are considered *closed* i.e. every edge is shared by exactly two faces. We will denote the faces of the polyhedron by $A, B, C, \ldots$, and the vertices by $u, v, w, \ldots$. An edge bounded by vertices u and v will be denoted by $\overline{uv}$. For any face A of a polyhedron, π_A will be the plane that contains the polygon A. A polyhedron is

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convex if its interior is a convex set i.e. for any points u and v interior to the polyhedron, the segment $\overline{uv}$ is interior. If a polyhedron is convex, then for any distinct faces A and B, all the points of A that are not in B are on the same side of the plane π_B. For any face A of a polyhedron we will denote by $N[A]$ the subcomplex of the polyhedron induced by A and the faces adjacent to A. $N[A]$ can be unfolded by cutting along its edges, except the edges that bound A. We will call this the *volcano unfolding* of $N[A]$. This unfolding blows out all the faces adjacent to A around A. In this section we will prove that if a polyhedron is convex, then the volcano unfolding of $N[A]$ is nonoverlapping. This generalizes the volcano unfolding of domes from Chapter 22 in [5]. (a *dome* is a polyhedron with a distinguished face called base, such that all nonbase faces are adjacent to the base.)

Theorem 1 *Let A be any face of a convex polyhedron. Then the volcano unfolding of $N[A]$ is nonoverlapping.*

Proof. Let's assume that there exists a convex polyhedron, such that for some face A, the volcano unfolding of $N[A]$ is overlapping. Let B and C be two faces adjacent to A that would overlap in the volcano unfolding of $N[A]$. If A, B , and C share a common vertex v, let α, β, and γ be the measures of the interior angles at v in faces A, B , and C respectively. Since the curvature of the polyhedron at v is positive, $\alpha + \beta + \gamma < 360°$, therefore B and C cannot overlap. (see Figure 1.)

Otherwise, let $\overline{uv}$ be the edge that faces A and B share,

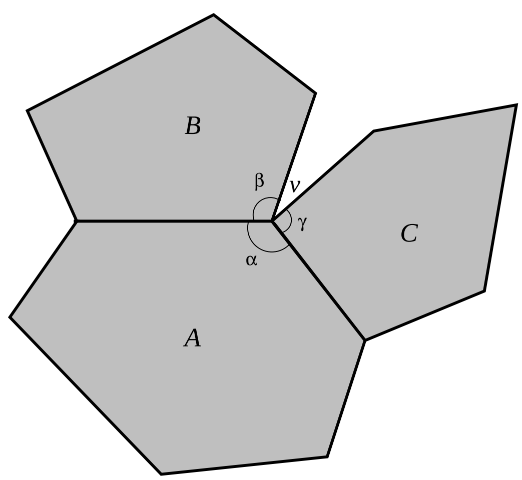

Figure 1: Faces B and C will not overlap when they share a vertex v.

and let $\overline{wx}$ be the edge that faces A and C share. Then u, v, w, and x are distinct and the quadrilateral $\square uvwx$ is convex. If lines $\overleftrightarrow{uv}$ and $\overleftrightarrow{wx}$ are parallel, then it is easy to see that faces B and C cannot overlap. (see Figure 2.) If lines $\overleftrightarrow{uv}$ and $\overleftrightarrow{wx}$ intersect, then let y be their point of intersection. (see the 3D Figure 3.) Since the planes

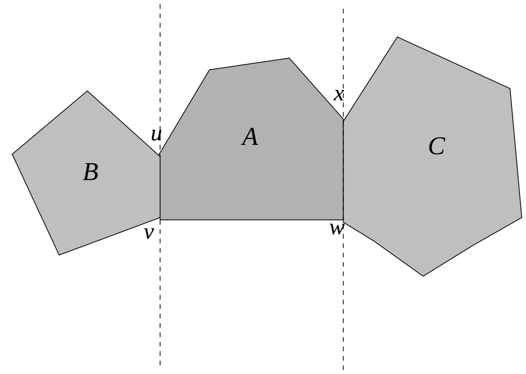

Figure 2: An unfolding of faces B and C on the plane π_A when $\overleftrightarrow{uv} \parallel \overleftrightarrow{wx}$.

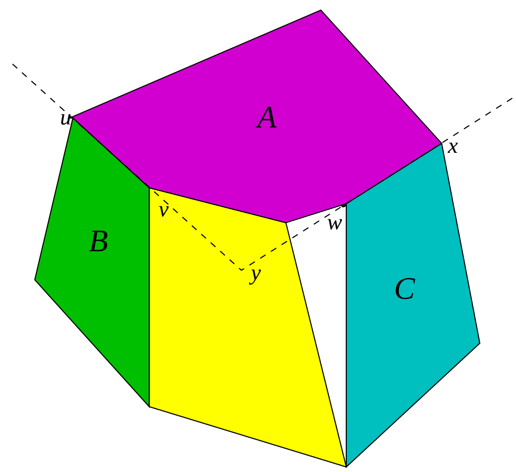

Figure 3: A polyhedron where lines $\overleftrightarrow{uv}$ and $\overleftrightarrow{wx}$ intersect.

that contain faces B and C have y in common, they must have a line in common. If $l = \pi_B \cap \pi_C$, let z be another point on l such that z is on the same side of the plane π_A as the polyhedron.(see Figure 4.) Since the polyhedron is convex, all the points on face B should be on the same side of the planes π_A and π_C, therefore they should be in the interior or on the boundary of angle $\angle uyz$. Similarly the points on face C of the polyhedron should be in the interior or on the boundary of angle $\angle wyz$.

Let α, β, and γ be the measures of the angles $\angle uyw$, $\angle uyz$, and $\angle wyz$. Then $\alpha+\beta+\gamma < 360°$. Let z' and z'' be the points that correspond to z in the unfolding of faces B and C on the plane π_A respectively. Since $\alpha + \beta + \gamma < 360°$, the interior of angles $\angle uyz'$, and $\angle wyz''$ are disjoint, therefore faces B and C cannot overlap. (see Figure 5.)

□

Since all the nonbase faces of a dome are adjacent to the base we obtain the following corollary previously proved in Chapter 22 of [5]:

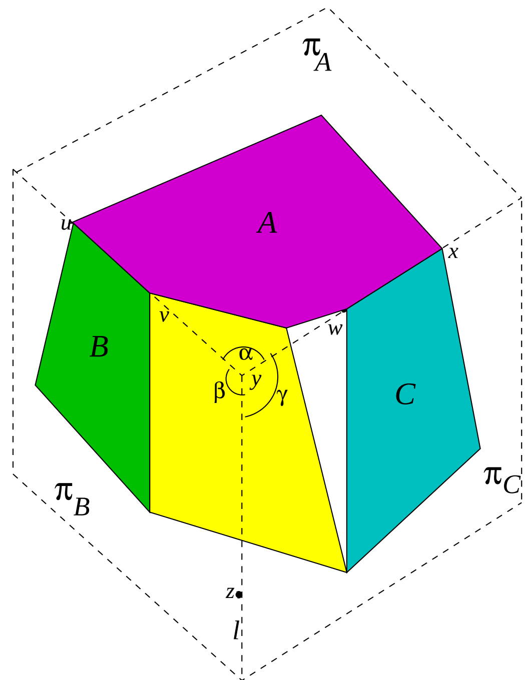

Figure 4: If the planes π_A, π_B and π_C meet at point y, then $\alpha + \beta + \gamma < 360°$.

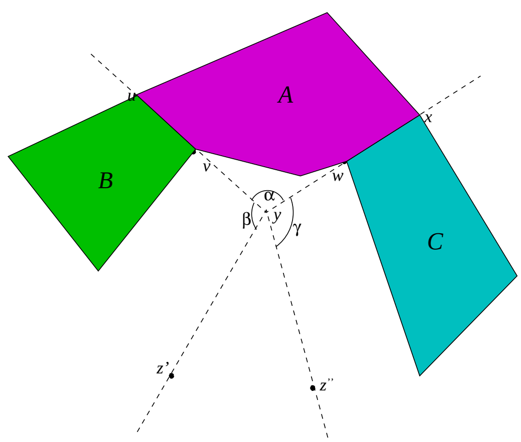

Figure 5: An unfolding of faces B and C on the plane π_A when lines $\overleftrightarrow{uv}$ and $\overleftrightarrow{wx}$ intersect.

Corollary 2 *All domes can be unfolded into one nonoverlapping net.*

Observation 1 *If the polyhedron is simplicial, then faces A, B, and C always meet at some vertex v, so the statement from Theorem 1 holds even if the polyhedron is not convex.*

3 The Domination Number of a Graph

Given a graph $G = (V, E)$, a set of vertices S is called a *dominating set* if every vertex in V is either in S or is adjacent to a vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. For example, sets $\{v, y\}$ and $\{u, x, z\}$ are domination sets for the graph in Figure 6. The domination number of this graph is 2. Determining whether a graph

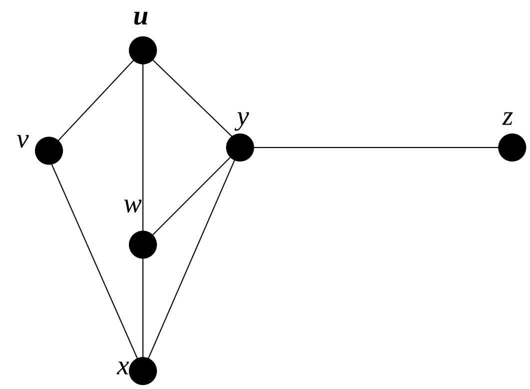

Figure 6: A graph with domination number 2.

has domination number at most k is NP-complete [7]. However, graphs with high minimum degree $\delta(G)$ have small domination numbers. In 1996 Reed [10] proved the following result:

Theorem 3 *If $G = (V, E)$ is a connected graph of order n and $\delta(G) \geq 3$, then $\gamma(G) \leq 3n/8$.*

The upper bound provided by Reed is sharp. Reed [10] also conjectured that if the graph is cubic (i.e. every vertex has degree 3), then the domination number is at most $\lceil n/3 \rceil$. Kostochka and Stodolsky [9] disproved this conjecture and found the following upper bound for cubic graphs:

Theorem 4 *If $G = (V, E)$ is a connected cubic graph of order $n > 8$, then $\gamma(G) \leq 4n/11$.*

4 Main Theorems

In this section we will use dominating sets in the dual graph of a polyhedron to find new upper bounds for the number of non-overlapping nets a convex polyhedron can be unfolded into. Here is our main result:

Theorem 5 *Every convex polyhedron with F faces can be unfolded into no more than $3F/8$ non-overlapping nets.*

Proof. Let's consider a convex polyhedron with F faces, and let $G = (V, E)$ be its dual graph. Then G has F vertices. Since every face of the polyhedron has at least three edges, $\delta(G) \geq 3$. Moreover G is connected, therefore by Theorem 3 we obtain $\gamma(G) \leq 3F/8$. Let $A_1, A_2, \ldots, A_k$ be the faces that correspond to a minimum dominating set for G. Then $k \leq 3F/8$. Then we can partition the set of faces of the polyhedron in

$S_1 \cup S_2 \cup \ldots \cup S_k$, where the sets $S_1, S_2, \ldots, S_k$ can be obtained from the following algorithm:

$S_0 := \{A_1, A_2, \ldots, A_k\}$
for $i := 1$ **to** k
$S_i := A_i \cup \{A : A$ is adjacent to A_i and $A \notin S_0\}$
$S_0 := S_0 \cup S_i$

Since $S_i \subseteq N[A_i]$, by Theorem 1 the volcano unfolding of the faces in S_i produces a non-overlapping net. This leads to an unfolding into $k \leq 3F/8$ non-overlapping nets. $\square$

If the polyhedron is simplicial, then the dual graph is cubic. An argument similar with the previous proof, combined with Theorem 4 provides an unfolding of the polyhedron into no more than $k \leq 4F/11$ non-overlapping nets. Note that according to Observation 1 the volcano unfolding produces non-overlapping nets even for nonconvex polyhedra. Therefore we obtain the following:

Theorem 6 *Every simplicial polyhedron with F faces can be unfolded into no more than $4F/11$ non-overlapping nets.*

5 Conclusion

The domination number in the dual graph of the polyhedron was a key ingredient in the proof of our main results. While the upper bound for the domination number provided by Reed's theorem is sharp, all the known examples of graphs that have the domination number equal to $3n/8$ are nonplanar. Establishing sharp upper bounds for the domination number in planar 3-connected graphs would improve the result from Theorem 5. A similar result for planar cubic graphs would improve the result from Theorem 6.

Acknowledgements
The fewest nets problem for convex polyhedra was discussed at the Reconnect workshop organized by the Center of Discrete Mathematics and Theoretical Computer Science (DIMACS) at Saint Mary's College, CA during the summer of 2004. The author is grateful to Joseph O'Rourke for his excellent series of lectures at that workshop, and for the numerous discussions he had with the author on the fewest nets problem during and after the workshop.

References

[1] D. Barnette Trees in Polyhedral Graphs, *Canad. J. Math.*, 18(1966), 731–736.

[2] M. Bern, E. Demaine, D. Epstein, E. Kuo, A. Mantler and J. Snoeyink Ununfoldable Polyhedra with Convex Faces, *Computational Geometry*, 24(2003), no. 2, 51–62.

[3] T. Biedl, E. Demaine, C. Duncan, R. Fleisher and S. Kobourov Tight Bounds on Maximal and Maximum Matchings, *Discrete Math.*, 285(2004), no. 1-3, 7–15.

[4] E. Demaine and J. O'Rourke Open Problems from CCCG 2003, *Proc. 16th Canad. Conf. Comput. Geom.*, Concordia(2007), 209–211.

[5] E. Demaine and J. O'Rourke *Geometric Folding Algorithms: Linkages, Origami and Polyhedra*, Cambridge University Press(2007).

[6] Albrecht Dürer *The Painter's Manual: A Manual Measurement of Lines, Areas and Solids by Means of Compass and Ruler Assembled by Albrecht Dürer for the Use of All Lovers of Art with Appropriate Illustrations Arranged to be Printed in the Year MDXXV*, Abaris Books, Inc., New York(1977). English Translation of *Unterweisung der Messung mit dem Zirkel und Richtscheit in Linien, Ebenen und Ganzen Körpern* (1525)

[7] M. Garey and D. Johnson *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W. H. Freeman(1979).

[8] T. Haynes, S. Hedetniemi and P. Slater *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc. New York(1998).

[9] A.V. Kostochka and B.Y. Stodolsky An Upper Bound on the Domination Number of n-Vertex Connected Cubic Graphs, *preprint*, (2007).

[10] B.A. Reed Paths, Stars and the Number Three, *Comb. Probab. Comput.*, 5(1996), no. 3, 277–295.

[11] G.C. Shephard Convex Polytopes with Convex Nets, *Mathematical Proceedings of the Cambridge Philosophical Society*, 78(1975), no. 3, 389–403.