Contraction and expansion of convex sets

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1 Introduction

Helly's theorem is one of the fundamental results in discrete geometry [9]. It states that if every $\leqslant d+1$ sets in a set system $\mathcal{S}$ of convex sets in $\mathbb{R}^d$ have non-empty intersection then all of the sets in $\mathcal{S}$ have non-empty intersection. Equivalently, if the entire family $\mathcal{S}$ has empty intersection, then there is a subset $\mathcal{S}' \subset \mathcal{S}$ (a *witness*) of size $\leqslant d+1$ which also has empty intersection. Over the years the basic Helly theorem has spawned numerous generalizations and variants [16]. These have the following local-global format: If every m members of a family have property P then the entire family has property P (or sometimes a weaker property P'). Equivalently, if the entire family has property P'^c then there is a witness subfamily of size m having the (possibly weaker) property P^c.

The conclusion of Helly's theorem fails, of course, if the sets in $\mathcal{S}$ are not convex; also if one changes the property "empty intersection" to notions of "small intersection". Nevertheless, we present Helly-type theorems that apply to cases of these sorts. We do so by allowing in the local-global transition not a weakening of the property P, but (arbitrarily slight!) changes in the sets themselves. We use a pair of operations, the *contraction* $C^{-\varepsilon}$ and *expansion* C^{ε} of a convex set C. For centrally symmetric convex sets these are simply homothetic scalings about the center (by factor $(1+\varepsilon)$ and $(1-\varepsilon)$ respectively), but for general convex sets the definitions are more complicated, and the operations appear to be new. The operations are continuous, i.e., for small $\varepsilon > 0$, the contraction $C^{-\varepsilon}$ and the expansion C^{ε} are close to C (in the Hausdorff metric). Our Helly type theorems are described below.

I. Finding a witness for small intersection Consider the case in which the given set system $\mathcal{S}$ consists of convex sets, however their intersection is not empty. In this case (as an analog to Helly's theorem) one may seek a witness of small cardinality $\mathcal{S}' \subseteq \mathcal{S}$ whose intersection is contained in the intersection of the sets of $\mathcal{S}$. It is not hard to verify that finite witnesses do not exist even for systems of convex sets in $\mathbb{R}^2$. For example, for any unit vector $u \in \mathbb{R}^2$, let C_u be the strip of width 2 consisting of vectors v with $\langle v, u\rangle \in [-1, 1]$. The intersection of the family $\mathcal{S} = \{C_u\}_u$ is the closed unit ball $\mathcal{B}$ centered at the origin, and any finite subset $\mathcal{S}'$ of this family has intersection which strictly includes $\mathcal{B}$. Namely, no finite witness for the intersection of $\mathcal{S}$ exists. We show:

Theorem 1 *Let $\varepsilon > 0$. Let $\mathcal{S} = \{C_i\}_i$ where each C_i is closed and convex in $\mathbb{R}^d$. There exists a subset $\mathcal{S}'$ of $\mathcal{S}$ of size at most $s(d, \varepsilon) = \frac{(cd)^d}{\varepsilon^{\frac{d}{2}}}$ such that $\cap_{C\in\mathcal{S}'} C^{-\varepsilon} \subseteq \cap_{C\in\mathcal{S}} C$. Here $c > 0$ is a universal constant.*

We remark that $s(d, \varepsilon)$ is essentially tight as a function of ε.

II. Sets that are not necessarily convex Now, consider the case in which the set system $\mathcal{S}$ does not consist of convex sets, but rather of sets that are the union of a bounded number of convex sets. Does the natural analog of Helly's theorem hold for such systems $\mathcal{S}$? Namely, if $\cap_{C\in\mathcal{S}} C$ is empty, is there a small witness $\mathcal{S}' \subseteq \mathcal{S}$ for this fact? As before, it is not hard to verify that the answer is no — even in the simplest case when all sets in $\mathcal{S}$ consist of the union of two convex sets in $\mathbb{R}^1$. For example, consider the family $\mathcal{S} = \{C_1, \ldots, C_{n-1}\}$ in which C_i is the closure of $[0, 1] \setminus [\frac{i-1}{n}, \frac{i+1}{n}]$. (Set difference is denoted "\".) Each C_i is the union of at most 2 closed intervals, and $\cap C_i = \phi$. However, any strict subfamily $\mathcal{S}'$ of $\mathcal{S}$ has non-empty intersection.

For real $f \geqslant 1$, a bounded convex set C is f-fat if the ratio between the radii of the minimum radius ball containing C and the maximum radius ball contained in C is at most f (see [5]; this is essentially inverse to the earlier definition [15]). If C is unbounded, C is not f-fat for any value f. For a set C consisting of the union of k convex sets $\{C_1, \ldots, C_k\}$ define $C^{-\varepsilon}$ to be $\cup_{i=1}^k C_i^{-\varepsilon}$. (Observe that the definition depends on the constituents C_i and not only on their union.) We show:

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Theorem 2 *Let $\varepsilon > 0$. Let $\mathcal{S} = \{C_i\}_i$ where each C_i is the union of at most k f-fat closed convex sets in $\mathbb{R}^d$. There exists a subset $\mathcal{S}'$ of $\mathcal{S}$ of size at most $s(k, d, \varepsilon, f) = (4k)^{k-1} \left(\frac{cdf^{k-1}}{\varepsilon^{k-\frac{1}{2}}} \right)^d$ such that $\cap_{C \in \mathcal{S}'} C^{-\varepsilon} \subseteq \cap_{C \in \mathcal{S}} C$. Here $c > 0$ is a universal constant.*

A few remarks are in place. First notice that if $\cap_{C \in \mathcal{S}} C = \phi$ then Theorem 2 states the existence of a small witness $\mathcal{S}'$ for empty intersection (extending Helly's theorem). Secondly, in $\mathbb{R}^1$ all bounded convex sets are 1-fat, so the fatness condition is not a restriction in $d = 1$. We conjecture that the fatness condition is unnecessary also in higher dimension (Conjecture 1). Finally, in $\mathbb{R}^1$ we are able to improve the value $s(k, 1, \varepsilon, 1)$ to approximately $(c/\varepsilon)^{k/2} \log^{k/2}(1/\varepsilon)$ for some constant $c > 0$. This value of $s(k, 1, \varepsilon, 1)$ can be shown to be essentially tight as a function of ε.

Conjecture 1 *Let $\mathcal{S} = \{C_i\}_i$ where each C_i is the union of at most k closed convex sets in $\mathbb{R}^d$. There exists a subset $\mathcal{S}'$ of $\mathcal{S}$ whose size depends only on k, d, and ε such that $\cap_{C \in \mathcal{S}'} C^{-\varepsilon} \subseteq \cap_{C \in \mathcal{S}} C$.*

1.1 Related work

To the best of our knowledge, these contraction and expansion operations for convex sets (except in the centrally symmetric case) have not previously been considered. (Minkowski sum with a unit ball is an entirely different operation as discussed in Section 2.) Also, we are not aware of other Helly-type theorems which apply to the case of non-empty intersection.

There is an interesting literature on Helly-type theorems for unions of convex sets. (For a nice survey on Helly type theorems in general see Wenger [16].) Let $\mathcal{C}_k^d$ be the family of all sets in $\mathbb{R}^d$ that are the union of at most k convex sets. The intersection of members in $\mathcal{C}_k^d$ are not necessarily in $\mathcal{C}_k^d$, and in general, as we have noted, subfamilies of $\mathcal{C}_k^d$ do not have finite Helly number (*i.e.,* there is not a finite witness for empty intersection). Nevertheless, it was shown independently by Matoušek [12] and Alon and Kalai [1] that if $\mathcal{S}$ is a finite subfamily of $\mathcal{C}_k^d$ such that the intersection of every subfamily of $\mathcal{S}$ is in $\mathcal{C}_k^d$, then $\mathcal{S}$ has finite Helly number. Let $\mathcal{K}_k^d$ be the family of all sets in $\mathbb{R}^d$ that are the union of at most k *pairwise disjoint* convex sets. As before $\mathcal{K}_k^d$ does not have finite Helly number. Helly type theorems for subfamilies $\mathcal{S}$ of $\mathcal{K}_k^d$ such that the intersection of every subfamily of $\mathcal{S}$ is in $\mathcal{K}_k^d$ have been studied. Grunbaum and Motzkin [8] showed that for $k = 2$ the Helly number of such $\mathcal{S}$ is $2(d + 1)$, and for general k conjectured it to be $k(d+1)$ (which is tight). The case $k = 3$ was proven by Larman [11], and the general case by Morris [13]. An elegant proof (based on the notion of LP-type problems) was presented by Amenta [3]. Differently from this literature, our results do not depend on a restriction on the intersections of subfamilies of $\mathcal{S}$.

1.2 Algorithmic motivation

Jie Gao and the authors of this work have recently used a variant of Theorem 2 in the design of an efficient approximation algorithm for clustering [6]. Roughly speaking, approximation algorithms lend themselves naturally to the notion of ε contraction and expansion — namely, in both cases a quantified *slackness* of ε is allowed.

More specifically, in [6] we prove a variant of Theorem 2 for sets that consist of unions of axis-parallel slabs. Our theorem is then applied in the design of an efficient dynamic data structure which manages the intersection of such sets. The data structure, in turn, is used as a key element in a $1 + \varepsilon$ approximation algorithm for the k-center clustering of incomplete data.

In general, Helly type theorems have found many algorithmic applications. One such example is the tight connection between the *generalized linear programming* (GLP) paradigm and Helly type theorems [14, 2]. It is plausible that the theorems and definitions presented in this paper will find algorithmic applications other than those presented in [6].

1.3 Proof Techniques

In Theorem 1, we wish to find a small witness for the intersection $A = \cap_i C_i$ of elements in $\mathcal{S} = \{C_i\}$. Namely, we are interested in a subset $\mathcal{S}'$ of $\mathcal{S}$ such that after its contraction, has intersection contained in A. Roughly speaking, we show that for any point x on the boundary of A, there exists a set $C_i \in \mathcal{S}$ such that the contraction $C_i^{-\varepsilon}$ of C_i does not include x together with a *significant* portion of the boundary of A. Finding such sets C_i iteratively, we are able to *cover* the boundary of A, resulting in the desired collection. As we are dealing with general convex sets, giving a precise quantification of our progress towards covering the boundary of A is the major technical difficulty in our proof. The proof of Theorem 1 somewhat resembles the proof of Dudley for convex shape approximation by a polytope with few vertices [4].

In Theorem 2 we wish to find a small witness for the intersection A of $\mathcal{S} = \{C_i\}$ when the sets C_i are not necessarily convex, rather they are the union of k convex sets. In a nutshell, the theorem is proven by induction on k, where Theorem 1 acts as the base case. The inductive step is non-trivial, and strongly uses the fatness f of the sets in C_i.

1.4 Organization

In the remainder of the body of this extended abstract (Section 2) we define the contraction and expansion of convex sets. Due to space limitations, the proofs of Theorems 1 and 2 are omitted and will appear in the full version of our paper.

2 The contraction and expansion of convex sets

We start with a few preliminary definitions and notation. Throughout this extended abstract all sets are subsets of $\mathbb{R}^d$ and the convex sets we consider are closed and bounded (convex sets in their full generality are considered in the full version of this abstract). S^{d-1} is the d dimensional unit sphere. For a set $A \subseteq \mathbb{R}^d$ and $c \in \mathbb{R}$ the set cA is $\{cx | x \in A\}$. Given two sets A and B the Minkowski sum $A + B$ is defined as the set $\{x + y | x \in A, y \in B\}$.

For a unit vector u, a u-hyperplane is a $d-1$ dimensional affine subspace perpendicular to u. A slab is the Minkowski sum of a hyperplane and a finite segment, and a u-slab is one bounded by u-hyperplanes. The u-slab of a set A is the closed u-slab of minimal width containing A; it is denoted $s_u(A)$ and its width is denoted $w_u(A)$.

Let us recall the standard definition of contraction and expansion for centrally symmetric sets. A convex set $C \subseteq \mathbb{R}^d$ is centrally symmetric iff it has a center p such that for any $x \in \mathbb{R}^d$: $p + x \in C$ iff $p - x \in C$. For a centrally symmetric convex set C let $\|x\|_C$ be the norm of x with respect to C: $\|x\|_C = \inf\{r > 0 \mid \frac{x-p}{r} + p \in C\}$. Now, for any $\varepsilon \geqslant -1$ define $C^\varepsilon = \{x \mid \|x\|_C \leqslant 1 + \varepsilon\}$. Namely, for positive ε the set C^ε is a *blown-up* version of C and is referred to as the expansion of C; and for negative ε the set C^ε is a *shrunken* version of C and is referred to as the contraction of C (see Figure 1). When ε tends to 0, the set C^ε *tends* to C. It is not hard to verify that C^ε is convex. Notice the distinction between C^ε and the Minkowski sum $C + \varepsilon\mathcal{B}$ (where $\mathcal{B}$ is the unit ball centered at the origin); the first definition commutes with affine linear transformations, the second does not.

We are now ready to define contraction and expansion of general convex sets C. More specifically, we would like to define the notion of a norm of $x \in \mathbb{R}^d$ with respect to C. First notice that we cannot use a direct analog to the definition for centrally symmetric bodies as a general convex set lacks a center point p. We thus consider an alternative definition to $\|x\|_C$ for centrally symmetric C which is independent of p. As we will see, such a definition generalizes naturally to convex sets which are not necessarily centrally symmetric.

Let C be centrally symmetric around the origin. Let u be any unit vector in $\mathbb{R}^d$. Consider the u-slab $s_u(C)$ of C. Clearly, it holds that $C \subseteq s_u(C)$. Moreover, it is not hard to verify that $C = \cap_{u \in S^{d-1}} s_u(C)$. Now consider C^ε. As C^ε is convex we have that $C^\varepsilon = \cap_u s_u(C^\varepsilon)$. However, it also holds that $s_u(C^\varepsilon) = s_u^\varepsilon(C) = (1+\varepsilon)s_u(C)$. Thus we conclude an alternative equivalent definition for contraction and expansion of symmetric convex sets: $C^\varepsilon = \cap_u s_u^\varepsilon(C)$. Namely, a definition which relies solely on the notion of contraction and expansion of slabs. This is the definition we would like to use for general convex sets.

Definition 2.1 (Contraction and Expansion) *Let C be a closed and bounded convex set and let ε be any real (positive, negative or zero). For a slab $s_u(C)$, let $s_u^\varepsilon(C)$ be defined by the standard definition of contraction and expansion for centrally symmetric sets stated in the beginning of Section 2. Let $C^\varepsilon = \cap_{u \in S^{d-1}} s_u^\varepsilon(C)$. For any $x \in \mathbb{R}^d$, let $\|x\|_C$ be $1+\varepsilon$ for the minimum ε such that $x \in C^\varepsilon$. The definition above is depicted in Figure 2.*

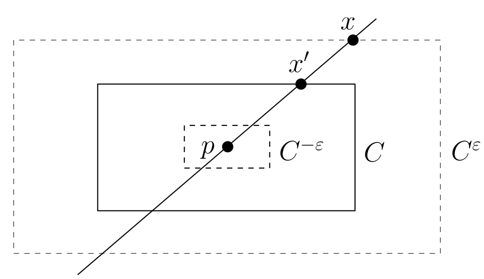

Figure 1: An illustration of the contraction and expansion of a centrally symetric convex set C. The set C with center p is given by a solid line. For positive ε, the expansion C^ε and the contraction $C^{-\varepsilon}$ of C are given by dashed lines. For the point $x \in C^\varepsilon$ we have that $x' = \frac{x-p}{1+\varepsilon}$ lies on the boundary of C.

Claim 2.1 (Convexity of $\|\cdot\|_C$) *For a convex set C, points x_1 and x_2 in $\mathbb{R}^d$, and $\lambda \in [0,1]$ it holds that $\|\lambda x_1 + (1-\lambda)x_2\|_C \leqslant \lambda\|x_1\|_C + (1-\lambda)\|x_2\|_C$.*

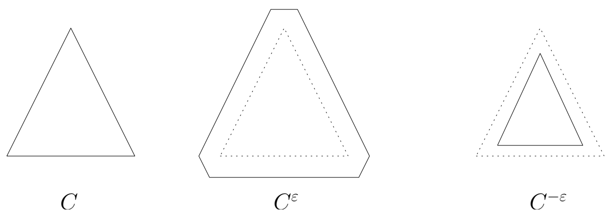

Figure 2: An illustration of Definition 2.1 applied to a triangle C (which is not centrally symmetric). Here $\varepsilon > 0$. In the presentation of C^ε and $C^{-\varepsilon}$, the set C is drawn with a dotted line and its expansion/contraction is presented as a solid line. Notice that the expansion C^ε of C is no longer a triangle.

Claim 2.2 (Continuity) *Let C be a convex set. Let $\varepsilon > 0$ then $(C^\varepsilon)^{-\frac{\varepsilon}{1+\varepsilon}} = C$. Let $\varepsilon > 0$ be sufficiently small (namely $\varepsilon < \frac{1}{d}$) then $C \subseteq (C^{-\varepsilon})^{\frac{d\varepsilon}{1-d\varepsilon}}$.*

Remark 2.1 *The bound on ε in the second part of Claim 2.2 may seem unnatural. However, it is necessary as for certain convex sets C the contraction $C^{-\varepsilon}$ is empty for $\varepsilon = \frac{c}{d}$ for some small constant $c > 0$. Thus, any expansion of $C^{-\varepsilon}$ in this case remains empty. For example, consider the d dimensional simplex C, it is not hard to verify that $x \in C$ implies that x is within distance approximately $\frac{1}{d}$ from ∂C. Thus, it is not hard to verify that, for $\varepsilon > \frac{2}{d}$, $C^{-\varepsilon} = \phi$.*

An alternative definition: One may consider an alternative definition for $\|x\|_C$ and C^ε in which the contraction and expansion are done with respect to a *center* point p in C. Namely, for $p \in C$ one can define $\|x\|_{(C,p)} = \inf\{r > 0 \mid \frac{x-p}{r} + p \in C\}$; and $C_p^\varepsilon = \{x \mid \|x\|_{(C,p)} \leqslant 1 + \varepsilon\}$. Clearly, as this definition depends strongly on the point p chosen, it is not equivalent to our original definition given in Definition 2.1. For this reason we prefer to use our original definition which depends solely on the set C. Nevertheless, for specific choices of p (such as the center of mass or the point derived from John's theorem [10]), one can find connections between our original definition and that given above.

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