Improved Upper Bounds on the Reflexivity of Point Sets

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Abstract

Given a set S of n points in the plane, the *reflexivity* of S, $\rho(S)$, is the minimum number of reflex vertices in a simple polygonalization of S. Arkin et al. [4] proved that $\rho(S) \leq \lceil n/2 \rceil$ for any set S, and conjectured that the tight upper bound is $\lfloor n/4 \rfloor$. We show that the reflexivity of any set of n points is at most $\frac{3}{7}n + O(1) \approx 0.4286n$. Using computer-aided abstract order type extension the upper bound can be further improved to $\frac{5}{12}n + O(1) \approx 0.4167n$.

1 Introduction

Given a set S of $n \geq 3$ points in the plane, a *polygonalization* of S is a simple polygon P whose vertices are the points of S. Throughout this paper we assume that the points are in *general position*, that is, no three of them are collinear. A vertex of a simple polygon is *reflex* if the (interior) angle of the polygon at that vertex is greater than π. We denote by $\rho(P)$ the number of reflex vertices of a polygon P. The *reflexivity* of a set of points S, $\rho(S)$, is the smallest number of reflex vertices any polygonalization of S must have. Further, we denote by $\rho(n)$ the maximum value $\rho(S)$, such that S is a set of n points. Table 1 lists $\rho(n)$ for $n \leq 10$. These values were verified using a computer [2, 4].

n	3	4	5	6	7	8	9	10
$\rho(n)$	0	1	1	2	2	2	3	3

Table 1: $\rho(n)$ for $n \leq 10$

The notion of reflexivity was suggested by Arkin et al. [4] as a measure for the "goodness" of a polygonalization of a set of points. They showed that $\lfloor n/4 \rfloor \leq \rho(n) \leq \lceil n/2 \rceil$ and conjectured that the lower bound is tight. Settling this conjecture is one of the open problems listed in *The Open Problems Project* [5]. We refer the reader to [4] for a more detailed discussion on the notion of reflexivity, its applications, and related problems.

Our main result is the following improvement for the upper bound of $\rho(n)$.

Theorem 1 $\rho(n) \leq 3\lfloor \frac{n-2}{7} \rfloor + 2$.

The result will be obtained by considering a slightly modified version of reflexivity, namely to force a given convex hull edge to be part of the polygonalization. The main ingredient is an iterative subdivision of the point set, together with a good polygonalization of sets of constant size. Theorem 1 then directly follows from Theorem 6 below.

Utilizing a computer-aided abstract order type extension we will further improve the upper bound to

Theorem 2 $\rho(n) \leq 5\lfloor \frac{n-2}{12} \rfloor + 4$.

2 Modified Reflexivity and Iterative Subdivision

Recall that the convex hull of a finite set S of points, $\mathcal{CH}(S)$, is composed of the boundary and the interior of a convex polygon. By abuse of notation we will make no distinction between that polygon and $\mathcal{CH}(S)$. To prove a stronger variant of Theorem 1 we first introduce some notation. Let S be a set of points and let e be an edge of of $\mathcal{CH}(S)$. We denote by $\rho_e(S)$ the minimum possible number of reflex vertices in a polygonalization P of S, such that e is an edge of P. Similarly, let $\bar{\rho}(S)$ be the maximum value of $\rho_e(S)$ taken over all the edges e of $\mathcal{CH}(S)$, and let $\bar{\rho}(n)$ be the maximum value of $\bar{\rho}(S)$ taken over all sets S of size n.

Obviously $\rho(n) \leq \bar{\rho}(n)$, so our goal is to derive good upper bounds for $\bar{\rho}(n)$. To this end we first provide a central lemma, which allows us to subdivide a point set in a way that we can consider the polygonalizations of the subsets rather independently.

Lemma 3 *Given an integer $k > 2$, a set S of $n > k$ points, and two points $p, q \in S$, such that pq is an edge of $\mathcal{CH}(S)$. Then, there exists a point $t \in S \setminus \{p, q\}$ and two sets $L, R \subset S$ such that:*

1. *$L \cup R = S$, $L \cap R = \{t\}$, $q \in R$, and $p \in L$;*
2. *The triangle $\triangle pqt$ contains no other points from S;*
3. *$\mathcal{CH}(R) \cap \mathcal{CH}(L) = \{t\}$;[1] and*
4. *$|R| = k$.*

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[1] Here $\mathcal{CH}(\cdot)$ denotes a convex set.

Proof. Assume, w.l.o.g., that p and q lie on the x-axis, such that p is to the left of q and all the remaining points are above the x-axis. Let t_1 be the point of S such that the angle $\angle t_1pq$ is the smallest. Let e_1 be the line determined by q and t_1, and let H_1 be the closed half-plane to right of e_1. Let S_1 be the subset of points of S contained in H_1. If $|S_1| > k$, then define $r_1 \in S_1 \setminus \{q, t_1\}$ to be the point creating the $(k-1)$st smallest angle $\angle r_1t_1q$, and denote by f_1 the line through t_1 and r_1. Otherwise, if $|S_1| \leq k$ let $f_1 = e_1$. Set $R_1 = \{q, t_1\} \cup \{p' \in S_1 | p' \text{ is to the right of } f_1\}$ and $L_1 = (S \setminus R_1) \cup \{t_1\}$. Note that r_1, if defined, is in L_1. We claim that t_1, R_1, and L_1 satisfy properties (1)–(3) of the lemma: (1) This property holds by the definition of R_1 and L_1; (2) By the choice of t_1 the triangle $\triangle pqt_1$ is empty; (3) All the points in R_1 are to the right of f_1, except for t_1 and possibly q. All the points in L_1 are to the left of f_1, except for t_1 and possibly r_1. However q and r_1 cannot both lie on f_1. If $|S_1| \geq k$, then we also have that $|R_1| = k$ (either by the choice of r_1 or because $|S_1| = k$, see Figure 1 for an illustration of the former case).

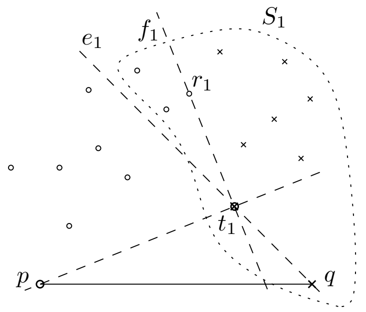

Figure 1: More than $k = 8$ points on or to the right of e_1. The points in R_1 and L_1 are marked by crosses and circles, respectively.

Suppose now that $|S_1| < k$. We define t_i, e_i, and S_i for $i > 1$ and $|S_{i-1}| < k$ recursively. Let t_i be the point that minimizes the angle $\angle t_ipq$ among the points in $L_{i-1} \setminus \{t_{i-1}\}$ (note that this set of points is not empty since $|S_{i-1}| < k$ and we show below that $S_1 \subset \cdots \subset S_{i-1}$). Let e_i be the line through q and t_i, let H_i be the closed half-plane to the right of e_i, and let S_i be the set of points contained in H_i. Similarly, we define r_i, f_i, R_i, and L_i. If $|S_i| > k$ define $r_i \in S_i \setminus \{q, t_i\}$ to be the point creating the $(k-1)$st smallest angle $\angle r_it_iq$, and denote by f_i the line through t_i and r_i. Otherwise, if $|S_i| \leq k$ set $f_i = e_i$. Set $R_i = \{q, t_i\} \cup \{p' \in S_i | p' \text{ is to the right of } f_i\}$ and $L_i = (S \setminus R_i) \cup \{t_i\}$. The existence of a point t and sets $R, L \subset S$ as required, will follow from the next claim.

Proposition 4 *Set $S_0 = \emptyset$. Then, for every $i \geq 1$ such that $|S_{i-1}| < k$, t_i, R_i, and L_i satisfy properties (1)–(3) of Lemma 3, and $S_{i-1} \subsetneq S_i$.*

Proof. By induction on i. For $i = 1$ the claim holds by the discussion above. Assume that $i > 1$ and $|S_{i-1}| < k$. Property (1) holds by the definition of R_i and L_i. The triangle $\triangle t_ipq$ is empty since: $\triangle t_{i-1}pq$ is empty; t_i is to the left of f_{i-1} and therefore $\triangle t_ipq$ does not contain any point from R_{i-1}; and by the choice of t_i. Thus, Property (2) holds. Property (3) clearly holds if $f_i = e_i$. Otherwise, if r_i is defined, denote by C_i the cone whose apex is at p and is bounded by the line through p and t_{i-1} and the line through p and t_i. By the choice of t_i all the points in C_i are in S_{i-1}. Since $|S_{i-1}| < k$ it follows that r_i is to the left of the line through p and t_i. Recall that r_i is to the right of e_i, since $r_i \in S_i$. Therefore, f_i is tangent to both $\mathcal{CH}(R_i)$ and $\mathcal{CH}(L_i)$ and separates them, except for the point t_i. Thus, Property (3) holds. Finally, since t_i is to the left of e_{i-1} we have $S_{i-1} \subseteq S_i$. However $t_i \in S_i \setminus S_{i-1}$, thus, $S_{i-1} \subsetneq S_i$. ☐

Since $|S_i| > |S_{i-1}|$ there is an integer j such that $|S_{j-1}| < k$ and $|S_j| \geq k$. It follows from Proposition 4 and the definition of R_j that t_j, R_j, and L_j satisfy the required properties. ☐

Note that Lemma 3 implies that pt is an edge of $\mathcal{CH}(L)$ and tq is an edge of $\mathcal{CH}(R)$, respectively. Using this fact we will apply the suggested subdivision in the next section in order to obtain our first main result.

3 A New Upper Bound

Figure 2 illustrates the subdivision obtained in the previous section. The idea to prove an upper bound on $\bar{\rho}(n)$ is to iteratively split a set into sub sets of constant size, to obtain good polygonalizations for these sets, and then to combine them according to Lemma 3. The base case is covered by the following result.

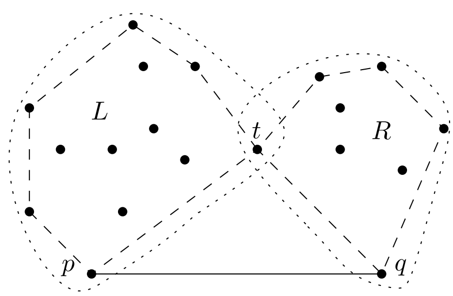

Figure 2: The subdivision of S guaranteed by Lemma 3

Lemma 5 *Let S be a set of at most 8 points in the plane. Then $\bar{\rho}(S) \leq 2$.*

Proof. The claim is clearly true for $n \leq 5$ since any point on $\mathcal{CH}(S)$ is a convex vertex of any polygonalization of S. For $6 \leq n \leq 8$ we prove the statement by a case-analysis over the size of the onion layers of S; see the full version of this paper for details. The correctness of the statement was also verified using a computer by checking all possible configurations of at most 8 points in general position. □

We are now ready for a first upper bound on $\bar{\rho}(n)$.

Theorem 6 $\bar{\rho}(n) \leq 3\lfloor \frac{n-2}{7} \rfloor + 2$.

Proof. We prove the claim by induction on n. For $n \leq 8$ we directly get the result from Lemma 5.

For $n > 8$ we apply Lemma 3 on the set S with $k = 8$ and some edge pq of $\mathcal{CH}(S)$, and obtain the point t and the subsets L and R. Now according to Lemma 5 there is a polygonalization of R containing the edge qt with at most two reflex vertices (note that qt is an edge of $\mathcal{CH}(R)$). By induction, L has a polygonalization containing the edge pt (that is an edge of $\mathcal{CH}(L)$) with at most $3\lfloor \frac{n-9}{7} \rfloor + 2 = 3\lfloor \frac{n-2}{7} \rfloor - 1$ reflex vertices. By removing the edge qt from the first polygonalization and the edge pt from the second, the remaining polygonal chains, along with the edge pq, form a proper polygonalization of S with at most $2+3\lfloor \frac{n-2}{7} \rfloor - 1 + 1 = 3\lfloor \frac{n-2}{7} \rfloor + 2$ reflex vertices (note that t may be a reflex vertex in the resulting polygon). □

4 Improving the Constant

Generalizing the approach used to prove Theorem 6 to arbitrary $k \geq 2$ we get

Corollary 7 *If for some $k \geq 2$ we have $\bar{\rho}(k) \leq l$, then $\rho(n) \leq (l+1)\lfloor \frac{n-2}{k-1} \rfloor + l \leq \frac{l+1}{k-1}n + l$.*

Improved bounds for $\bar{\rho}(n)$ for small, constant values of n thus yield a better bound on the reflexivity of arbitrarily large sets of points. From Lemma 5 together with an extension to $n = 9, 10$ by using the point set order type data base [2] we observe that $\bar{\rho}(n) = \rho(n)$ for $n \leq 10$, see Table 1. Therefore our next goal is to determine good bounds on $\bar{\rho}(n)$ for $n \geq 11$.

Lemma 3 implies that $\bar{\rho}(n) \leq \bar{\rho}(n-k+1) + \bar{\rho}(k) + 1$ for any $2 \leq k < n$. Using the values of Table 1 for $k = 3$ and $k = 8$ we get

$$\bar{\rho}(n) \leq \bar{\rho}(n-2) + 1 \quad (1)$$
$$\bar{\rho}(n) \leq \bar{\rho}(n-7) + 3$$

Applying these two relations we obtain the upper bounds on $\bar{\rho}(n)$ shown in Table 2 with an exception for $n = 13$.

n	11	12	13	14	15	16
$\rho(n)$	3	3..4	3..4	4..5	4..5	4..6
$\bar{\rho}(n)$	4	4	4	4..5	4..5	4..6

Table 2: $\rho(n)$ and $\bar{\rho}(n)$ for $n = 11 \ldots 15$

By using the point set order type data base for $n = 11$ points it turned out that $\rho(11) = 3$ whereas $\bar{\rho}(11) = 4$. Interestingly only for 36 of the 2 334 512 907 existing order types required the best polygonalization 4 reflex vertices. All these sets had a triangular convex hull, and only for one (out of three) convex hull edge e we obtained $\rho_e(S) = 4$. This has to be seen in contrast to the worst case examples for $\rho(S)$ obtained in [4], which are so-called double circles. There half of the vertices are on the convex hull, and the remaining vertices form a second onion layer, each point lying close to the middle of one edge of the convex hull.

The examples providing $\bar{\rho}(11) = 4$ together with Equation 1 imply $\bar{\rho}(12) = 4$. So we will have to look for values of $k > 12$ in order to benefit from Corollary 7. Thus we aim to show that $\bar{\rho}(13) = 4$.

From Equation 1 we already know that $\bar{\rho}(13) \leq 5$. So assume that there exists a set S, $|S| = 13$, with $\bar{\rho}(S) = 5$. By Equation 1 S contains a subset S' of 11 points with $\bar{\rho}(S') = 4$. We now apply abstract order type extension, which is a tool that can be used to generate all (abstract) point sets containing a given class of sets of smaller cardinality, see [3] for details. Applying this method to the 36 sets of $n = 11$ points which require 4 reflex vertices, we obtain all sets S for $n = 13$ which might require 5 reflex vertices. Our computations show that all obtained sets contain a polygonalization with at most 4 reflex vertices, and we conclude that $\bar{\rho}(13) = 4$.

By Corollary 7 we therefore get

Corollary 8 $\bar{\rho}(n) \leq 5\lfloor \frac{n-2}{12} \rfloor + 4$

which implies Theorem 2. Obviously determining $\bar{\rho}(n)$ for $n \geq 14$ could further improve the constant of Corollary 8, and we leave this for future research.

5 Further Results

Conjecture 3.4 in [4] states that $\rho(n) = \lfloor \frac{n}{4} \rfloor$. Considering the values for $\rho(n)$ in Tables 1 and 2 the conjecture has to be modified to

Conjecture 1 $\lfloor \frac{n}{4} \rfloor \leq \rho(n) \leq \lceil \frac{n}{4} \rceil$.

After establishing the existence of a polygonalization with few reflex vertices we describe an efficient way to find one.

Theorem 9 *Given a set of n points S and two points $p, q \in S$ such that pq is an edge of $\mathcal{CH}(S)$, a polygonalization P of S such that pq is an edge of P and $\rho(P) \leq 5\lfloor \frac{n-2}{12} \rfloor + 4$ can be found in $O(n \log n)$ time.*

The proof of this claim and an algorithm to generate the required subdivisions can be found in the full version of this paper.

Corollary 7 makes the following conjecture interesting.

Conjecture 2 *There is a constant c_0 such that $\bar{\rho}(n) \leq \rho(n) + c_0$.*

Note that the stronger statement that $\bar{\rho}(S) \leq \rho(S) + O(1)$ for any set S might also hold.

Conjecture 2, if true, would mean that it is possible to get arbitrarily close to the best possible linear upper bound by checking only finitely many small cases. In other words, suppose the conjecture holds and c is a constant such that $\rho(n) \leq cn$. Then, for any $\epsilon > 0$ there is $k = k(\epsilon)$ such that if we verify that $\rho(k) \leq ck$, then for $n > k$ we have $\rho(n) \leq (c+\epsilon)n + O(1)$. Indeed, k large enough such that $\frac{ck+c_0+1}{k-1} \leq (c+\epsilon)$ holds, would do. Moreover, the discussion above is still valid if we replace c_0 in Conjecture 2 by some function $f(n)$ such that $f(n) \in o(n)$.

Conjecture 2 is true when we consider reflexivity in the presence of *Steiner points*. Following the notation of [4], a *Steiner point* is a point $q \notin S$ that may be added to S in order to improve some structure. For example, we define the *Steiner reflexivity* of S, $\rho'(S)$, to be the minimum number of reflex vertices of any simple polygon with vertex set $V \supseteq S$. Similarly, $\rho'(n) = \max_{|S|=n} \rho'(S)$. The (stronger statement) of Conjecture 2 can be easily proved if we allow Steiner points.

Lemma 10 *Let S be a set of n points and let pq be an edge in $\mathcal{CH}(S)$. Then, there are points p', q' (inside $\mathcal{CH}(S)$) such that $S \cup \{p', q'\}$ has a polygonalization containing the edge pq and having at most $\rho(S) + 1$ reflex vertices.*

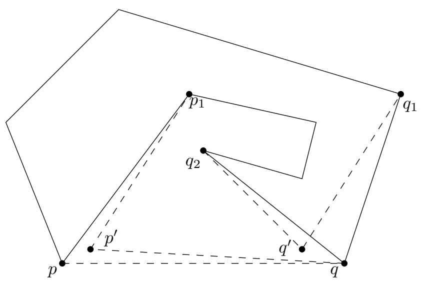

Figure 3: An illustration for the proof of Lemma 10

Proof. We assume, w.l.o.g., that the fixed edge pq is horizontal, p is left to q, and the remaining points $S \setminus \{p, q\}$ are above the line through p and q. Let P be a polygonalization of S, such that P does not contain the edge pq. We show that P can be modified into a polygonalization P' of a set $V \supset S$ such that P' contains the edge pq and $\rho(P') \leq \rho(P) + 1$.

Let p_1 be the counter-clockwise neighbor of p in P, and let q_1 and q_2 be the counter-clockwise and clockwise neighbors of q in P, respectively. Fix p' slightly to the right and above p, and q' slightly to the left and above q. Now by replacing the chain $q_1\,q\,q_2$ with the chain $q_1\,q'\,q_2$, and the edge $p\,p_1$ with the chain $p\,q\,p'\,p_1$, one obtains the desired polygonalization P' (see Figure 3 for an illustration). Note that the only reflex vertex that might be introduced in these steps is p'. □

As before, this implies that we can get arbitrarily close to any linear upper bound on $\rho'(n)$ by checking only finitely many small cases. Note that it is important here that the Steiner points we add lie inside the convex hull of the original set of points.

6 Discussion and Open Problems

We showed that for every set S of n points in general position in the plane there is a polygonalization of S with at most $5\lfloor \frac{n-2}{12} \rfloor + 4$ reflex vertices, and such a polygonalization can be found in $O(n \log n)$ time. The basic idea of the proof is that by Lemma 3 we can subdivide S into some fixed-size parts and use a stronger result on each of these parts. It would be interesting to find other applications of the subdivision suggested in Lemma 3.

It is challenging to determine the structure of sets maximizing the reflexivity for fixed cardinality. On the one hand we have the sets used in [4] to provide the bound of $\rho(n) \geq \lfloor \frac{n}{4} \rfloor$, which have half of their vertices on the boundary of the convex hull. This so-called double circle configuration is also conjectured to minimize the number of triangulations [1], and therefore seems to be a promising extremal example, supporting Conjecture 1. On the other hand all maximizing examples for $\bar{\rho}(11)$ have a triangular convex hull, so it could be that for larger cardinality $\bar{\rho}(n)$ is more than a constant factor larger than $\rho(n)$, contradicting Conjecture 2.

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