# On the Number of Empty Pseudo-Triangles in Point Sets

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## Abstract

We analyze the minimum and maximum number of empty pseudo-triangles defined by any planar point set. We consider the cases where the three convex vertices are fixed and where they are not fixed. Furthermore, the pseudo-triangles must either be star-shaped or can be arbitrary.


## 1 Introduction

Counting empty convex $k$-gons in planar point sets is a classic problem in combinatorial geometry that goes back to Erdős. In particular, he asked for the smallest number $N(k)$ such that any set $P$ of at least $N(k)$ points contains the vertex set of a convex $k$-gon whose interior does not contain any point of $P$. A related question asks for the minimum number of empty convex $k$-gons any set of points must contain.

These questions naturally generalize to other polygons that do not need to be convex. In particular, we are interested in *pseudo-triangles*, which are simple polygons that have exactly three convex vertices with internal angles less than $\pi$, see Figure 1. A pseudo-triangle is the "most reflex" polygon possible and can be considered the natural counterpart of convex polygons. In this note we study the number of empty pseudo-triangles that are contained in planar point sets. Since not every pseudo-triangle is *star-shaped* (see, for example, the rightmost pseudo-triangle in Figure 1) we consider this question both for star-shaped and general pseudo-triangles. Somewhat surprisingly, the answers differ by orders of magnitude.

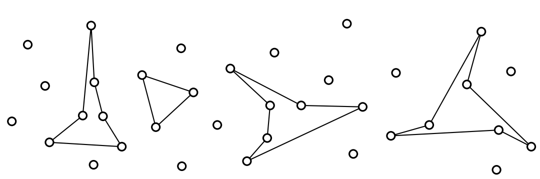

Figure 1: Empty pseudo-triangles in a point set.

Concerning the number of empty convex polygons, it was shown in [3, 4, 5] that for any set $P$ of $n$ points in general position, there are $\Omega(n^2)$ subsets of three, four, five, and six points that form empty convex triangles, quadrilaterals, pentagons, and hexagons. These bounds are tight. Furthermore, there are arbitrarily large sets of points that do not contain any empty convex heptagon. Trivially, for any constant $k$, the maximum number of empty convex $k$-gons is $\Theta(n^k)$, which is obtained by taking $n$ points in convex position.

In this note we consider the minimum and maximum number of pseudo-triangles that are contained in a set $P$ of $n$ points. If we do not require the pseudo-triangles to be empty of points in $P$, then we can have exponentially many of them. For example, place one point $p_0$ at the origin and all other $n-1$ points on the lower left quarter of the circle $(x-1)^2 + (y-1)^2 = 1$. Then $p_0$ together with any subset of $P \setminus \{p_0\}$ of size $\geq 2$ forms a pseudo-triangle, so there are at least $2^{n-1}-n$ of them. The minimum number of pseudo-triangles, empty or not, is cubic (e.g., for points in convex position, see Theorem 7).

Hence we concentrate on empty pseudo-triangles. First we assume that a triangle $\triangle uvw$ is given together with a set $P$ of $n$ points inside it. We analyze the number of empty pseudo-triangles that have $u$, $v$, and $w$ as the convex vertices, and all other vertices must be points of $P$. In this setting we have four combinatorial questions, namely the *minimum* and *maximum* number of empty *general* or *star-shaped* pseudo-triangles. We give tight upper and lower bounds for each question; our results are summarized in the table below. Observe that the (asymptotic) number of empty pseudo-triangles in the general case can be quadratic or cubic, depending on the point set, but in the star-shaped case we always get the same, quadratic bound.

| Section 2 | general | star-shaped |
|---|---|---|
| minimum | $\Theta(n^2)$ | $\Theta(n^2)$ |
| maximum | $\Theta(n^3)$ | $\Theta(n^2)$ |

If the convex vertices of the empty pseudo-triangles are not specified, then we get the same four questions that should be settled with an upper and a lower bound. Most of the ideas used before can be extended. We obtain the results summarized in the table below.

| Section 3 | general | star-shaped |
|---|---|---|
| minimum | $\Theta(n^3)$ | $\Theta(n^3)$ |
| maximum | $\Theta(n^6)$ | $\Theta(n^5)$ |

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## 2 Pseudo-triangles with given corners

Let $u$, $v$, and $w$ be three points in the plane, and let $P$ be a set of $n$ points inside the triangle $\triangle uvw$. We assume that no three points of $P \cup \{u, v, w\}$ lie on a line. Any pseudo-triangle with $u$, $v$, and $w$ as the convex vertices has a concave chain between $u$ and $v$, denoted $C(u, v)$, and also concave chains $C(v, w)$ and $C(w, u)$. For an empty pseudo-triangle, any point of $P$ lies on one of the three chains or in one of the three convex polygons $C(u, v) \cup \overline{vu}$, $C(v, w) \cup \overline{wv}$, and $C(w, u) \cup \overline{uw}$. If a point lies in $C(u, v) \cup \overline{vu}$, then we say that the point is *excluded by* the chain $C(u, v)$ (and analogous terminology is used for exclusion by $C(v, w)$ and $C(w, u)$). We denote the line that passes through two points $p$ and $q$ by $\ell(p, q)$.

**Observation 1** *If $P$ is partitioned into $P_{u,v} \cup P_{v,w} \cup P_{w,u}$, such that all points in $P_{x,y}$ lie on $C(x, y)$ or are excluded by $C(x, y)$ (with $x, y \in \{u, v, w\}$ and $x \neq y$), then at most one empty pseudo-triangle exists that has this partition. It is formed by the edges of the convex hulls of $P_{u,v} \cup \{u, v\}$, $P_{v,w} \cup \{v, w\}$, and $P_{w,u} \cup \{w, u\}$, where $\overline{uv}$, $\overline{vw}$, and $\overline{wu}$ are removed.*

### 2.1 General pseudo-triangles

**Theorem 1** *Given three points $u, v, w$ and a set $P$ of $n$ points inside $\triangle uvw$, the maximum number of empty pseudo-triangles with $u, v, w$ as the convex vertices is $\Theta(n^3)$.*

**Proof.** The lower bound is an easy construction, see Figure 2. To prove the upper bound, let $p_i$, $p_j$, and

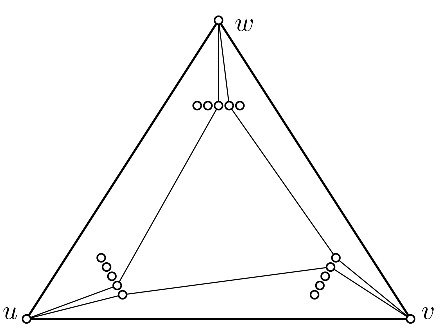


Figure 2: Sometimes $\Omega(n^3)$ pseudo-triangles.

$p_k$ be any three points of $P$. We analyze the number of pseudo-triangles such that edge $\overline{up_i}$ is on the chain $C(u, v)$, edge $\overline{vp_j}$ is on the chain $C(v, w)$, and edge $\overline{wp_k}$ is on the chain $C(w, u)$. Clearly, if any of the three edges intersect, then no pseudo-triangle exists of this type. Otherwise, we extend the edges $\overline{up_i}$, $\overline{vp_j}$, and $\overline{vp_j}$ in a special way.

Assume first that $\ell(u, p_i)$ is below $\ell(v, p_j) \cap \ell(w, p_k)$. Then we let point $q_i = \ell(u, p_i) \cap \ell(v, p_j)$, point $q_j = \ell(v, p_j) \cap \ell(w, p_k)$, and point $q_k = \ell(w, p_k) \cap \ell(u, p_i)$,

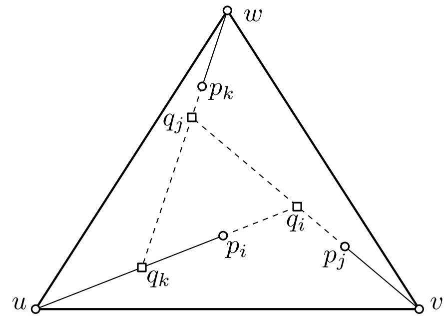


Figure 3: Always $O(n^3)$ pseudo-triangles, first case.

see Figure 3. If $\triangle q_i q_j q_k$ contains points of $P$, then no pseudo-triangle exists of this type: Either the pseudo-triangle would not be empty, or the concavity of one of the chains is compromised. Furthermore, all points in $\triangle uvq_i$ must be excluded via the chain $C(u, v)$, all points in $\triangle vwq_j$ must be excluded via the chain $C(v, w)$, and all points in $\triangle wuq_k$ must be excluded via the chain $C(w, u)$. This fully defines the partition as in Observation 1, so we count at most one pseudo-triangle.

Next assume that $\ell(u, p_i)$ is above $\ell(v, p_j) \cap \ell(w, p_k)$; note that due to the specification of $p_i$, $p_j$, and $p_k$, this case is not symmetric to the previous one. The argument, however, is still analogous. This time we let point $q_i = \ell(u, p_i) \cap \ell(w, p_k)$, point $q_j = \ell(v, p_j) \cap \ell(u, p_i)$, and point $q_k = \ell(w, p_k) \cap \ell(v, p_j)$, see Figure 4. If

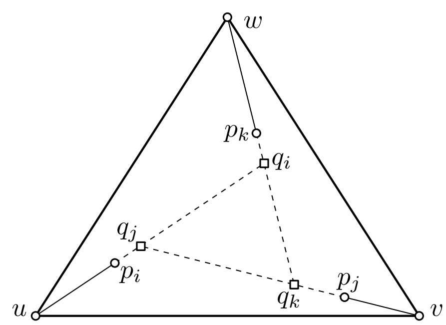


Figure 4: Always $O(n^3)$ pseudo-triangles, second case.

$\triangle q_i q_j q_k$ contains points of $P$, then no pseudo-triangle exists of this type: Either the pseudo-triangle would not be empty, or the concavity of one of the chains is compromised, or the chains would intersect. The rest of the argument is exactly as in the previous case, so we again count at most one pseudo-triangle.

All remaining cases ($\ell(u, p_i)$ contains $\ell(v, p_j) \cap \ell(w, p_k)$, or some chain(s) do not contain points of $P$) are straightforward to analyze. Since there are $6 \cdot \binom{n}{3}$ choices for $p_i$, $p_j$, and $p_k$, the upper bound follows. $\square$

**Theorem 2** *Given three points $u, v, w$ and a set $P$ of $n$ points inside $\triangle uvw$, the minimum number of empty pseudo-triangles with $u, v, w$ as the convex vertices is $\Theta(n^2)$ (assuming non-degeneracy).*

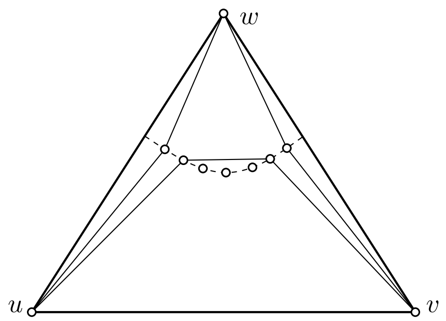


Figure 5: Sometimes $O(n^2)$ pseudo-triangles.

**Proof.** This time we begin with the upper bound, which is an easy construction shown in Figure 5. All points of $P$ are placed on a circular arc centered at $w$ (any minor perturbation can be used to remove this degeneracy). There are only $O(n)$ choices for the chain $C(v, w)$, only $O(n)$ choices of the chain $C(w, u)$, and given these choices, the chain $C(u, v)$ is completely specified since we only count empty pseudo-triangles. The quadratic upper bound follows.

Next we prove the lower bound. We need to show that any set $P$ gives $\Omega(n^2)$ pseudo-triangles. Let $(p_i, p_j)$ be any pair of points from $P$. If $\ell(p_i, p_j)$ does not intersect $\overline{uv}$ then we assign the pair to $\overline{uv}$. Similarly, if $\ell(p_i, p_j)$ does not intersect $\overline{vw}$ then we assign the pair to $\overline{vw}$, and if $\ell(p_i, p_j)$ does not intersect $\overline{wu}$ then we assign the pair to $\overline{wu}$. Due to non-degeneracy, we assign each pair from $P$ to exactly one side of $\triangle uvw$.

By symmetry and the pigeon-hole principle we may assume that $\overline{uv}$ is assigned $\Omega(n^2)$ pairs of points. Make each pair ordered so that $\overline{up_i}, \overline{p_ip_j}, \overline{p_jv}$ is a concave chain that does not self-intersect. Let $q = \ell(u, p_i) \cap \ell(v, p_j)$, see Figure 6, and let $q'$ be infinitesimally above $q$. Then an empty pseudo-triangle exists that excludes the points of $P \cap \triangle uvq'$ via the chain $C(u, v)$, that excludes the points of $P \cap \triangle vwq'$ via the chain $C(v, w)$, and that excludes the points of $P \cap \triangle wuq'$ via the chain $C(w, u)$. Furthermore, $\overline{up_i}$ and $\overline{vp_j}$ are the extreme edges of $C(u, v)$. Hence, for any other pair $(p_k, p_l)$ assigned to $\overline{uv}$ we get a different pseudo-triangle. Since $\Omega(n^2)$ edges were assigned to $\overline{uv}$, there are $\Omega(n^2)$ different pseudo-triangles. □

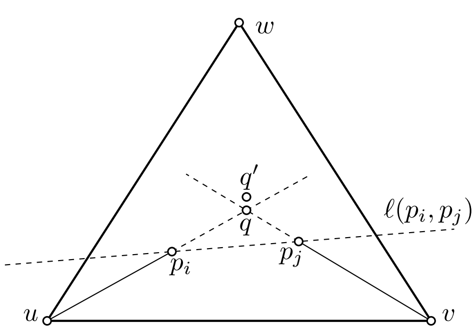


Figure 6: Always $\Omega(n^2)$ pseudo-triangles.

The non-degeneracy assumption in the theorem above is essential. If all points of $P$ lie on a line that also passes through $w$, for example, then there are only $O(n)$ different empty pseudo-triangles.

## 2.2 Star-shaped pseudo-triangles

It turns out that the number of empty star-shaped pseudo-triangles does not vary with $P$, asymptotically, and is always quadratic. The lower-bound proof of Theorem 2 generates only star-shaped pseudo-triangles, because $q'$ is always in the kernel. So the *minimum* number of empty shar-shaped pseudo-triangles is $\Omega(n^2)$. It remains to prove that the *maximum* number of empty star-shaped pseudo-triangles is also $O(n^2)$.

**Lemma 3** *Given three points $u, v, w$ and a set $P$ of $n$ points inside $\triangle uvw$, the maximum number of empty star-shaped pseudo-triangles with $u, v, w$ as the convex vertices is $O(n^2)$.*

**Proof.** Consider the set of $3n$ lines defined by one point of $P$ and one point of $\{u, v, w\}$, see Figure 7. These lines form an arrangement of quadratic size. Let $q$ be any point inside a cell of the arrangement. If we assume that $q$ is in the kernel of the empty pseudo-triangle, then the pseudo-triangle is completely determined: all points of $P \cap \triangle uvq$ are excluded via the chain $C(u, v)$, and the analogous statement holds for $P \cap \triangle vwq$ and $P \cap \triangle wuq$. By the choice of lines, no matter where $q$ lies in its cell, the subsets $P \cap \triangle uvq$, $P \cap \triangle vwq$, and $P \cap \triangle wuq$ are the same. Since there are $O(n^2)$ combinatorially distinct positions for $q$, the lemma follows. □

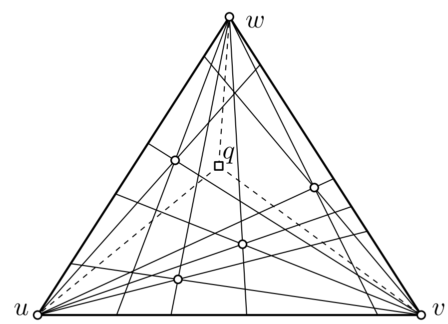


Figure 7: Always $O(n^2)$ star-shaped pseudo-triangles.

**Corollary 4** *Given three points $u, v, w$ and a set $P$ of $n$ points inside $\triangle uvw$, the minimum and maximum number of empty star-shaped pseudo-triangles with $u, v, w$ as the convex vertices is $\Theta(n^2)$.*

## 3 Pseudo-triangles in point sets

We discuss the case where the convex vertices are not given in advance. The results in the previous section give rise to some easy results for this case.

**Theorem 5** *Given a set $P$ of $n$ points in the plane, the maximum number of empty pseudo-triangles is $\Theta(n^6)$.*

**Proof.** The upper bound follows by taking all triples of $P$ as $u$, $v$, and $w$ and using the result of Theorem 1.

The lower bound follows by taking the construction of Theorem 1, using only $n/2$ points inside $\triangle uvw$, and replacing $u$, $v$, and $w$ by $n/6$ points each. $w$ is replaced by $n/6$ points on a horizontal line, very closely spaced and at $w$. Similarly, $u$ and $v$ are replaced by $n/6$ points each, and on lines that make angles of 60 degrees (for $v$) and $-60$ degrees (for $u$) with the $x$-axis. □

The same proof adaptations give the result on the maximum number of star-shaped pseudo-triangles. We simply state the result:

**Theorem 6** *Given a set $P$ of $n$ points in the plane, the maximum number of empty star-shaped pseudo-triangles is $\Theta(n^5)$.*

We will give one more result, namely that any point set gives $\Omega(n^3)$ different empty star-shaped pseudo-triangles. It completes the study of the number of empty pseudo-triangles, since a point set in convex position gives only $O(n^3)$ different pseudo-triangles.

**Theorem 7** *Given a set $P$ of $n$ points in the plane, the minimum number of empty pseudo-triangles (star-shaped or arbitrary) is $\Theta(n^3)$.*

**Proof.** We only need to prove two results: there is a set of $n$ points that gives $O(n^3)$ empty not necessarily star-shaped pseudo-triangles, and any point set gives $\Omega(n^3)$ empty star-shaped pseudo-triangles. For the former claim, simply take a set of $n$ points in convex position. For the latter claim, take any three points $p_i$, $p_j$, and $p_k$ of $P$. We will show that an empty star-shaped pseudo-triangle exists with $p_i$, $p_j$, and $p_k$ as the convex vertices. Take any point $q$ in the interior of $\triangle p_ip_jp_k$, and so that $\overline{p_iq}$, $\overline{p_jq}$, and $\overline{p_kq}$ do not contain any point of $P$. Consider the pseudo-triangle that excludes any points of $P \cap \triangle p_ip_jq$ via chain $C(p_i, p_j)$, any points of $P \cap \triangle p_jp_kq$ via chain $C(p_j, p_k)$, and any points of $P \cap \triangle p_kp_iq$ via chain $C(p_k, p_i)$. Clearly this gives a pseudo-triangle with $p_i$, $p_j$, and $p_k$ as the convex vertices and $q$ in the kernel. All $\binom{n}{3}$ choices of $p_i$, $p_j$, and $p_k$ give different pseudo-triangles. □

## 4 Conclusions

We have given tight bounds on the minimum and maximum number of empty pseudo-triangles that either must be star-shaped or may be arbitrary. The constructions and proofs are simple and elegant. An open question is whether pseudo-triangles that are 9-gons are necessary to have $\Omega(n^3)$ and $\Omega(n^6)$ empty pseudo-triangles in Theorems 1 and 5, or that smaller complexity pseudo-triangles can also be used.

### Acknowledgements

The results presented in Theorems 1, 5, and 7 were independently obtained by Oswin Aichholzer and Alexander Pilz [2] during the runtime analysis of an experimental project accompanying [1].

### References

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