

Unfolding Polyhedra via Cut-Tree Truncation

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Abstract

We prove that an infinite class of convex polyhedra, produced by restricted vertex truncations, always unfold without overlap. The class includes the “domes,” providing a simpler proof that these unfold without overlap.

1 Introduction

It is a long unresolved question whether or not every convex polyhedron may be cut along edges and unfolded to a single, non-overlapping polygon [DO05, DO07]. (Henceforth, we use “unfolding” to mean this type of “edge unfolding.”) Although an extensive exploration by Schlickerieder [Sch97] failed to find either an algorithm or a counterexample, only a few narrow classes of polyhedra are known to be unfoldable: prisms, prismoids, and “domes” [DO07, Chap. 22]. Here we slightly extend the latter class, via a new proof.

An unfolding is determined by a polyhedron P and a cut-tree T that spans the vertices of P . We define a property of a pair (P, T) that permits derivation of a new pair (P', T') , where P' has more vertices than P , such that (P', T') determines a non-overlapping unfolding.

We say that an unfolding has the *empty sector property* if the circular sector in the unfolding defined by each edge incident to a leaf vertex x of the cut-tree T is empty. The circular sector of an edge is the arc swept out between the two sides of its unfolding. The angle of the arc will be exactly the angle deficit of the leaf vertex; the radius of the arc will be the length of the edge. If y is the parent of x in T , then this sector is defined by xy_1 and xy_2 , where y_1 and y_2 are the two unfolded images of y , with sector angle the curvature at x . See Fig. 1(b).

Many unfoldings have the empty sector property. For example, take any tetrahedron, and a Y cut-tree, the star from some vertex. This clearly produces an unfolding with the empty sector property. Every unfolding of the cube has the empty sector property.

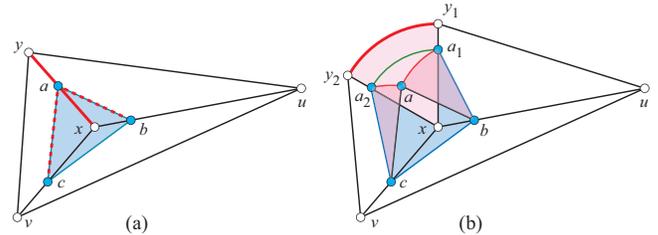


Figure 1: Sector nesting for triangle truncation. (a) Vertex x of P is truncated to $\triangle abc$ in P' ; (b) The unfolding of P and of P' in the vicinity of x .

Let x be a leaf of the cut-tree T . A *leaf truncation* is a truncation of the vertex x of the polyhedron P to P' , and a corresponding alteration of the cut-tree to T' , so that, if y is the parent of x in T , and $(a, p_1, p_2, \dots, p_k)$ is the polygon resulting from the truncation, with $a \in xy$, then, T' follows the “claw” $(y, a, p_1, p_2, \dots, p_i)$ and $(y, a, p_k, p_{k-1}, \dots, p_{i+1})$, leaving some edge $p_i p_{i+1}$ uncut. For example, if x has degree 3 in P (as in Fig. 1(a)), then the truncation polygon is a triangle $\triangle abc$ ($bc = p_1 p_2$), and the claw becomes the Y $\{ya, ab, ac\}$. We call this a *degree-3 leaf truncation*, to which the theorem below is restricted.

2 Main Theorem

Theorem 1 *If a non-overlapping unfolding of a polyhedron P via a cut-tree T has the empty sector property, then the cut-tree T' produced by a degree-3 leaf truncation (a) unfolds P' without overlap, and (b) has the empty sector property.*

Proof: Let x be a degree-3 leaf of the cut-tree T with parent y , and let x have incident edges $\{yx, ux, vx\}$. Let the truncation of x produce $\triangle abc$ on the truncated polyhedron P' , with $a \in yx$, $b \in ux$, $c \in vx$, as illustrated in Fig. 1(a). The cut-tree T' includes the Y $\{ya, ab, ac\}$; the edge bc is uncut.

We now compare the unfolding of P and of P' in the vicinity of x . The curvature α at x is the angle gap in the layout of the triangles $\triangle xyu$, $\triangle xuv$, $\triangle xvy$ around x in the unfolding of P . A key observation is that the unfolding of P' leaves these triangles fixed in the same position but truncated. The edges ub and vc remain uncut, and so maintain the relative positions of the triangles. The truncation triangle $\triangle abc$ is affixed at

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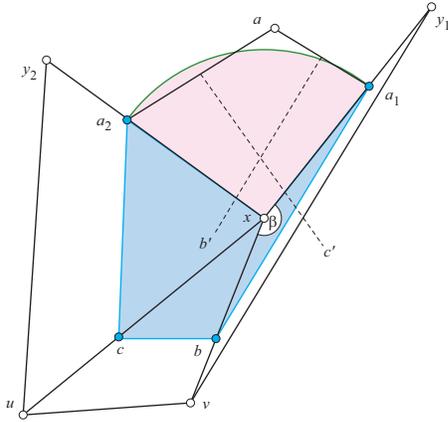


Figure 2: If a is outside the sector, then the positions of b and c are reversed to b' and c' .

edge bc in the P' unfolding. (See Fig. 1(b).) Our goal is to prove that $\triangle abc$ remains inside sector (x, a_1, a_2) , and therefore inside the enclosing sector (x, y_1, y_2) .

The position of a , the tip of $\triangle abc$ in the unfolding of P' , is the point where the rotations of ba_1 and ca_2 meet. Let $\beta = \angle a_1xb$ and $\gamma = \angle a_2xc$. Then, as is evident in Fig. 2, the rotation of a_1 must start inside sector (x, a_1, a_2) in a neighborhood of a_1 , because the convexity of β places b right of the line through xa_1 . And similarly for a_2 . However, it is conceivable that this rotation ends with a outside the sector. Suppose it does, as in the figure. Then it must be that b lies on the perpendicular bisector of a_1a , because a is the rotation of a_1 about b at a distance $|ba_1|$; and similarly c lies on the perpendicular bisector of a_2a . But these bisectors must cross inside the sector, each passing to the “wrong side” of x , because a is outside the sector (if a were on the arc a_1a_2 , then the bisector would pass through x). This interchanges the presumed positions of b and c , a contradiction.

To be more precise on this interchange claim, let L be the line containing bc . Then the perpendicular bisector of a_1a meets L in a point b' that is left of the point c' at which the bisector of a_2a meets L . But we labeled the vertices so that b is right of c on L .

Although our figures use $\alpha < \pi$, nothing changes with $\alpha \geq \pi$. Therefore, we have established the sector nesting property claimed in the lemma. \square

Fig. 3 shows that the Y-split choice made in the definition of degree-3 truncation is necessary.

Corollary 2 Any polyhedron P derived by repeated degree-3 leaf truncations from an initial polyhedron P_0 and cut-tree T_0 with the empty sector property, unfolds without overlap via the derived cut-tree T .

See Fig. 4 for an example.

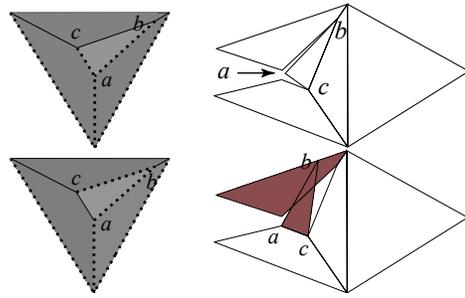


Figure 3: Including the edge bc from Fig. 1 can lead to overlap.

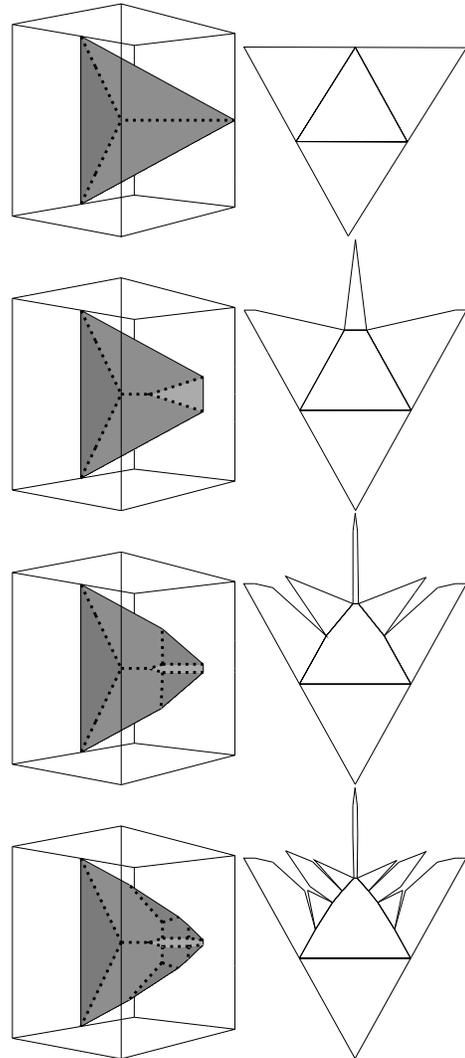


Figure 4: A series of degree-3 truncations refining one corner of a tetrahedron.

3 A counterexample for a truncation of degree > 3

This theorem does not hold in its most general form for truncation of leaves of degree $\delta > 3$. This negative result is demonstrated by a pyramid P with a five-sided wide and thin base, shown in Fig. 5. The cut tree T of P has the empty sector property at every vertex. Note that T follows the ‘spine’ of P from base to apex.

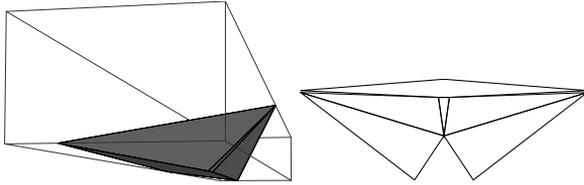


Figure 5: (a) Pyramid P with five-sided base. (b) The pyramid, unfolded.

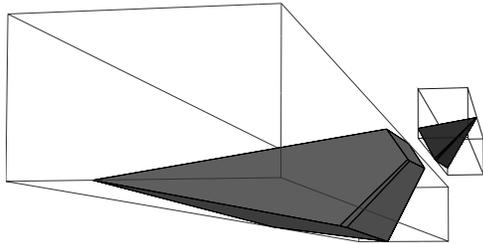


Figure 6: The truncated pyramid P' with tip removed.

We truncate P to P' , replacing the apex of P with a new five-sided face f (Fig. 6). Retaining all previous cut edges of T and extending T' to include four of the new edges of f , we find that there is no edge of f at which it may be joined to the unfolding of P' without conflict (Fig. 7). This establishes that Theorem 1 cannot be extended to arbitrary k .

4 Cutting to achieve degree- k vertices

As shown above, truncating a vertex of degree > 3 may introduce conflict in the unfolding. This prohibits the removal of such vertices through truncation but does not prohibit the creation of higher-degree vertices.

Let a be a degree-3 leaf node of cut-tree T , and b the parent of a . Perform a leaf-node truncation to produce T_ϵ , where the truncation triangle cuts ab a small distance ϵ from b . By Theorem 1, the unfolding produced by T_ϵ strictly avoids overlap. Note that b is “nearly” degree-4; see Fig. 8. Letting $\epsilon \rightarrow 0$ changes b to truly degree-4 without causing overlap. This process can be repeated on any degree-3 leaf to increase the degree of its immediate parent. Note that this argument results in every ϵ -edge being part of T and therefore cut, which is essential.

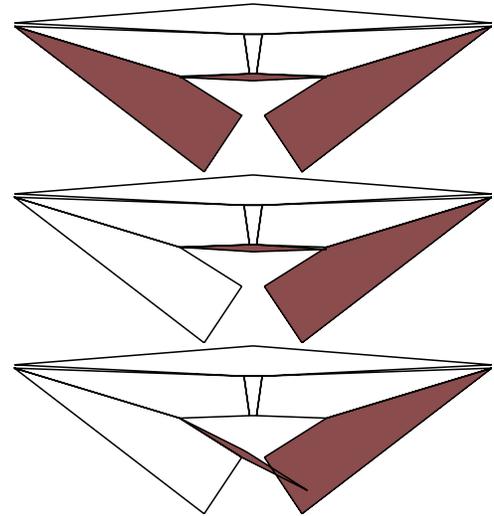


Figure 7: All five attachments of the new face f to P' (two symmetric and not shown) lead to overlap.

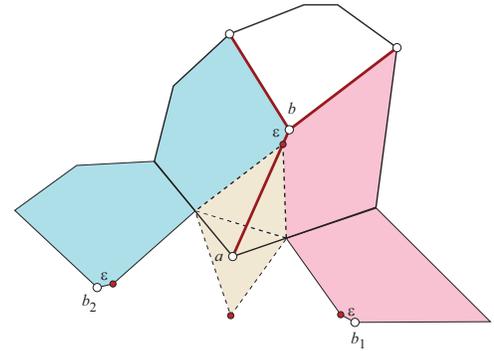


Figure 8: When $\epsilon \rightarrow 0$, node b of T_ϵ becomes degree-4.

5 Empty sector property essential

We now show that the empty sector property is necessary for Theorem 1. Consider the convex cap C in Fig. 9(a), a subset of a larger convex polyhedron not shown. C is cut by some cut-graph T to the unfolding shown in Fig. 9(b). This local subset of the unfolding avoids overlap but does not have the empty sector property.

In Fig. 9(c) the degree-3 vertex x in Fig. 9(a) has been truncated, replacing it with a triangular face and extending T with new edges. This immediately introduces a conflict into the unfolding, seen in Fig. 9(d).

6 Polyhedra achievable by degree-3 leaf truncation

It is of interest to know which shapes are achievable by degree-3 leaf truncation, for all these shapes are edge-unfoldable. The class depends on the initial (P_0, T_0) pair. Starting from a pyramid leads to the class of “domes,” which were defined and proved to be unfold-

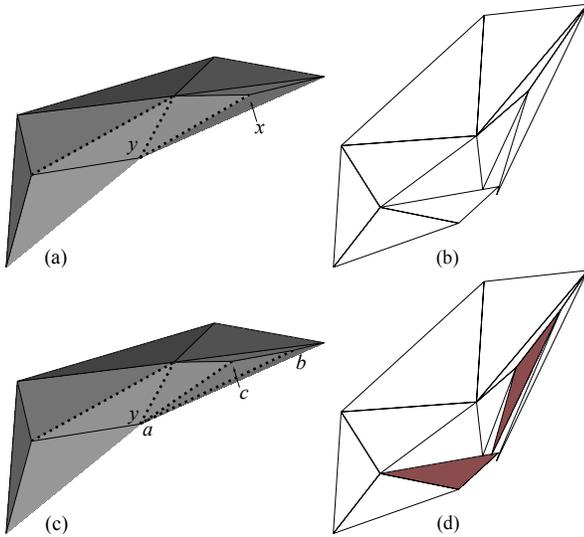


Figure 9: Without the empty sector property, degree-3 vertex truncation cannot guarantee non-overlap.

able without overlap in [DO07, Sec. 22.5]. A *dome* is a convex polyhedron with a distinguished *base* polygon B , and the property that every nonbase face shares an edge with B . We will exclude from the proof below the special dome we call a *wedge*, whose base B is a trapezoid.

Theorem 3 *Starting from a pyramid P_0 and cut-tree T_0 the star of edges incident to the apex a of P_0 , the polyhedra achievable via degree-3 leaf truncation are all domes. And conversely, every dome (except possibly a wedge) can be realized by a series of degree-3 leaf truncations from some pyramid P_0 .*

Proof: *Pyramid \Rightarrow Dome.* Let i index the i -th truncation. The leaf nodes of T_i all lie on B_i . Degree-3 truncation is defined only on leaf nodes of T , therefore every truncation will intersect B_i and create two new vertices which also lie on B_i . Thus every face created will have at least two vertices on B (Cf. Fig. 10), ensuring that P_i is a dome.

Dome \Rightarrow Pyramid. Let D be a dome. The dual graph of its faces, one node per nonbase face, with two nodes connected if their faces share an edge, is planar, and moreover outerplanar (every node on the exterior face), because every face is incident to the base B . It is a well-known result of graph theory that every outerplanar graph has at least two nodes of degree 2. Such nodes correspond to triangle faces of D .

The plan is to extend the two faces incident to the nonbase edges of a triangle T of D . The extension of these faces may not meet, but because the quoted result guarantees two such triangles, at least one must have its face extensions meet. This can be seen as follows. Let a, b, c be three consecutive edges of the base polygon B .

If the extension of a and c over b does not meet, then the turn angle of these three edges must be $\geq \pi$. Since the total turn angle around B is 2π , any other such extension must turn $\leq \pi$. A wedge constitutes the $=\pi$ case, so with that excluded, if one extension diverges then the other extension converges.

So let T be the triangle whose face extensions meet. This results in a new dome D_1 with one fewer vertex on its base face B_1 . Note the highest vertex of D_1 is the same as that in D , as it could not have been altered by the face extensions. Continuing in this manner, we arrive at a dome D_k with a triangular base B_k . Clearly D_k must be a tetrahedron.

Now we view D_k as P_0 , and view the reverse of a face extension as a degree-3 truncation. This shows that D can be derived from some particular tetrahedron D_k , establishing the claim. \square

This provides an alternative and arguably simpler proof that domes unfold without overlap. (That the excluded wedges unfold without overlap is straightforward.) Moreover, Corollary 2 reaches a class of polyhedra larger than domes, for the starting P_0 and T_0 just need the empty sector property. For example, starting from a non-overlapping unfolding of a cube or a dodecahedron will lead to polyhedra that are not domes.

7 Conclusion

We have shown through a restricted series of vertex truncations that certain classes of polyhedra can be unfolded without overlap. One such class is the “domes”. Although we have demonstrated several directions in which these results cannot be extended, we feel that there are a number of related avenues still worthy of further investigation. For example, perhaps there are geometric (curvature?) conditions at a leaf vertex of degree > 3 that permit truncations without overlap.

In general, we believe that the notion of deriving a non-overlapping unfolding from a simpler non-overlapping unfolding deserves further study.

References

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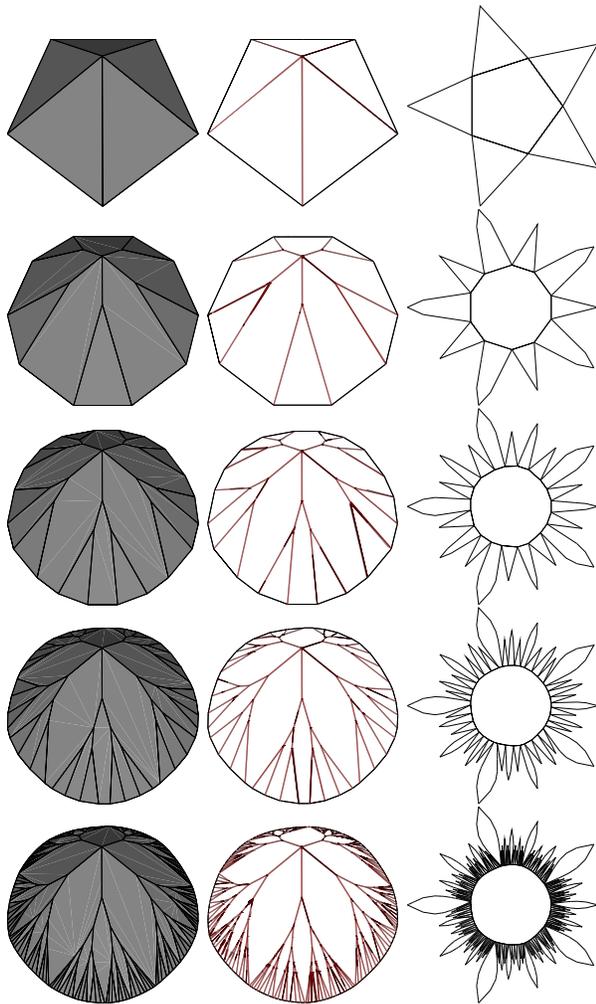


Figure 10: Progressive truncation of $P_i \rightarrow P_{i+1}$ retains the empty sector property and unfoldability. Left: top view of polyhedron; middle: cut tree; right: unfolding.