Realizations of hexagonal graph representations

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Abstract

A hexagonal graph representation is a description of a drawing of a planar graph with maximum degree 6 such every edge segment knows its direction vector, but coordinates of vertices are unknown. Moreover, the angles are such that each face and each vertex of the graph could be realized locally.

In this paper, we study realizations of hexagonal graph representations, i.e., the problem of finding vertex coordinates such that all edge directions are realized. We provide three results, sadly all negative: First, not all hexagonal graph representations have a realization. Secondly, even if one has a realization, its size may be exponential in the number of vertices. Finally, testing whether it has a realization is NP-hard.

1 Background

In his seminal paper on orthogonal drawings of planar graphs, Tamassia [10] obtained bend-minimal drawings by breaking the problem into two parts: In the first part, he finds an orthogonal representation, which is a description of a graph drawing that fixes the number of bends and all angles, but not the actual drawing. Finding a bend-minimal orthogonal representation can be done via minimum cost flow in polynomial time. In the second part, Tamassia then converts such an orthogonal representation into an actual orthogonal drawing, by assigning suitable lengths to all segments of edges such that the resulting drawing has no crossing. This so-called *orthogonal compaction* can always be done by "cutting off an ear" from any face that is not a rectangle; once all faces are rectangles a suitable set of edge lengths can be found easily. (Other authors studied orthogonal compaction with additional constraints; see Patrignani's paper [7] and the references therein.)

Tamassia also considered k-gonal drawings, which are drawings in which edges are routed with segments that have slope $i \cdot \pi/k$ for some $i \in \{0, \dots, k-1\}$. For $k = 3$, k-gonal drawings are also called *hexagonal drawings.* With only minor changes, his first phase works for k slopes as well, i.e., we can find a bend-minimum k-gonal representation of a planar graph in polynomial time.

Tamassia does not address the issue of realizing such a k-gonal representation. Some later works deal with this, but they either only work for triangular graphs (all interior faces are triangles) [2], or series-parallel graphs [5], or prove NP-hardness for larger k [5].

In this paper, we study the problem of realizing hexagonal (i.e., 3-gonal) representations. Not all 3-gonal representations can be realized; see Section 3. Even if a realization exists, we show in Section 4 that it may require very large edge length. Finally, we show in Section 5 that testing whether a hexagonal representation has a realization is NP-hard. It was already known that testing whether a planar graph with given angles (in fact, a 12-gonal representation) has a realization is NP-hard [5]. Our contribution is hence to reduce k needed for NP-hardness. In fact, the complexity of realizing hexagonal representations is explicitly mentioned in [11].

2 Definitions

We assume familiarity with graph theory and especially planar graphs. A *(multi-line) graph drawing* of a planar graph assigns a point to each vertex, and a contiguous sequence of line segments connecting the points of the endpoints to each edge. A *bend* is the place where the drawing of an edge changes slope. We will often identify the graph-theoretic concept (vertex, edge) with the geometric element that represents it in its drawing (point, sequence of line segments). For the drawing to be valid, it must be *planar*, i.e., no two vertices coincide, and edges are disjoint except at common endpoints.

We say that a graph drawing is a *k-gonal drawing* if it has at most k different slopes among the segments of edges. A 2-gonal drawing is usually called *orthogonal drawing*; a 3-gonal drawing is called a *hexagonal drawing*. Such a drawing is called a *grid drawing* if vertices and bends are placed at the intersection points of a rectangular/hexagonal grid.

A *k-gonal representation* of a planar graph gives for each edge the number of segments, and for each segment a direction (the slope as a vector from one endpoint to the other.) At most k different slopes are used. Moreover, the edge directions are locally correct for each vertex and face, i.e., around each vertex the edge segments (sorted by slope) correspond to the planar embedding, and around each face with d edge segments, the edge directions form angles that sum up to $(d-2)\pi$ degrees.

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A *k-gonal representation* is called *orthogonal representation* for $k = 2$ and *hexagonal representation* for $k = 3$. A k-gonal drawing *realizes* a k-gonal representation if all the edge directions coincide.

3 Existence

In this section, we give examples of hexagonal representations that cannot be realized. Consider the representation in Figure 1(a) (only the edge slopes should be considered, as this is a representation, not a drawing.) The graph is essentially a "wheel"-graph, except that one spoke of the wheel has been extended by another edge. All but one interior face are triangles whose angles are all $\pi/3$, so they are equilateral, so all spokes have the same length. However, the second edge of the extended spoke then must have length 0 in any hexagonal drawing, which is not allowed. A similar argument shows that the hexagonal representation in Figure 1(b) also has no realization, and in addition it has maximum degree 3.

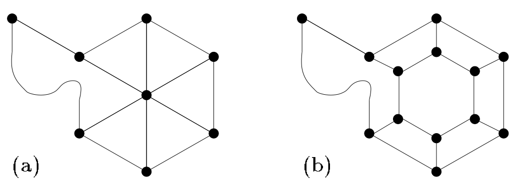

Figure 1: Hexagonal representations that have no realizations.

Theorem 1 *There exist hexagonal representations without realizations, even if the graph has maximum degree 3.*

Note that this theorem holds even if the realization is not required to be a grid drawing; all that we have used is that edges are drawn with non-zero length.

4 Exponential edge length

In this section, we show that even if a hexagonal representation can be realized, it may take a lot of edge length (hence area) to do so. In fact, rather than giving one specific hexagonal representation, we give a planar embedded graph and show that for any bend-minimal hexagonal drawing of this graph, the maximum edge length must be exponential.

Define graph G_i, $i \geq 1$, as follows (see also Figure 2):

- G_1 is a triangle with vertices v_0^0, v_0^1 and v_0^2.
- For any $i > 1$, let G_i be the graph obtained from G_{i-1} by adding three more vertices v_i^0, v_i^1, v_i^2 in the outer-face, and adding a 6-cycle $v_{i-1}^0 - v_i^0 - v_{i-1}^1 - v_i^1 - v_{i-1}^2 - v_i^2 - v_{i-1}^0$ that becomes the new outer-face.

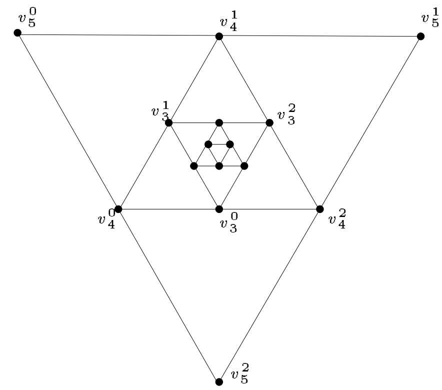

Figure 2: A graph for which any hexagonal drawing without bends requires exponential area.

Note that G_i has $3i$ vertices. Let Γ_N be a planar hexagonal drawing of G_N (for some large N) that has no bends and respects the planar embedding. Γ_N induces a hexagonal drawing Γ_i of G_i for each $1 \leq i \leq N$ which also has no bends. We first show that the angles in Γ_N must be as in Figure 2.

Lemma 2 *For any $i > 1$, the outerface of G_i is drawn as an equilateral triangle in Γ_i with corners at v_i^j, $j = 0, 1, 2$.*

Proof. Recall that the outer-face of G_i is a 6-cycle $v_{i-1}^0 - v_i^0 - v_{i-1}^1 - v_i^1 - v_{i-1}^2 - v_i^2$. By the fixed planar embedding, v_{i-1}^j (for $j = 0, 1, 2$) has two more edges on the inside. Hence the interior angle at v_{i-1}^j is at least π. The interior angle at v_i^j ($j = 1, 2, 3$) is at least $\pi/3$ since we have a hexagonal drawing. If any of these angles is bigger than their minimum, then the 6-cycle has angle-sum $> 4\pi$, and hence cannot be drawn as a 6-gon, so one of the edges on it must have a bend. But Γ_i has no bend. So the angles are all at their minimum, which means that the 6-cycle is drawn as a triangle with all angles $\pi/3$, i.e., it is equilateral. □

Γ_1 also must be an equilateral triangle, for any other hexagonal drawing of a 3-cycle has at least one bend . From this, and the details of the above proof, we deduce that all angles must be as in Figure 2. With that, we can now easily prove the claim about the edge length.

Lemma 3 *Assume the edges of Γ_1 have length α. Then the edges on the outerface of Γ_i have length $2^{i-2}\alpha$ for $i > 1$.*

Proof. We proceed by induction on i. The base case ($i = 2$) holds because v_1^0, v_2^0 and v_1^1 form an equilateral triangle, so edges (v_1^0, v_2^0) and (v_2^0, v_1^1) have length $\alpha = 2^{i-2}\alpha$ as well, and similarly one shows the claim for all other edges on the outer-face of Γ_2. For $i > 2$, we know that the path $v_{i-1}^0, v_{i-2}^1, v_{i-1}^1$ forms a straight line segment, and by induction it has length $2 \cdot 2^{i-3}\alpha = 2^{i-2}\alpha$. This path forms an equilateral triangle with the edges (v^0_{i-1}, v^0_i) and (v^0_i, v^1_{i-1}), so they too must have length $2^{i-2}\alpha$, which proves the claim. □

Theorem 4 *There exists a planar graph G with maximum degree 4 such that any hexagonal grid drawing of G that respects the planar embedding and has the minimum number of bends has maximum edge length at least $2^{n/3-2}$ times the minimum edge length.*

The above theorem requires that the planar embedding is fixed, but even if we allow arbitrary planar embeddings, we still get an exponential lower bound. Note that G_N would be 3-connected if the degree-2 vertices were replaced by edges. Therefore, G_N has only one combinatorial embedding, and the only way to draw it differently is to choose a different outer-face. For any choice of the outer-face, there is a still a copy of $G_{N/2}$ (possibly with three edges subdivided) that is drawn in the correct planar embedding. Hence there is an edge of length $2^{N/2-2}\alpha = 2^{n/6-2}\alpha$.

5 NP-hardness

In this section, we prove that testing whether a given hexagonal representation can be realized is NP-hard. Note that we do not know whether this problem is NP (and since exponential edge length may be required, it is non-trivial to argue that it should be.)

We know of multiple ways of proving NP-hardness of this problem, but have chosen here one that relates to another graph drawing problem, namely, orthogonal compaction with minimal maximum edge length. In this problem we are given an orthogonal representation and a constant K, and we want to find an orthogonal drawing that realizes the representation and has edge length at most K. Patrignani [7] showed that this problem is NP-hard even for a connected graph, and even if the graph has no bends at all.

For our reduction, it will be convenient to represent hexagonal representations and drawings on a grid that has vertical, horizontal, and diagonal grid lines; note that this is the same as a hexagonal grid after a horizontal shear. The horizontal and vertical grid-lines are used to copy (with some duplication) the given orthogonal representation, while the diagonals are used to enforce distance constraints.

So given an orthogonal representation of a connected graph, we construct a hexagonal representation as follows:[1] Replace every vertex by a rectangle with a diagonal in it. The diagonal forces the rectangle to be a square in any hexagonal drawing. Replace each edge (v, w) by a rectangle connecting the squares for v and w. There are no diagonals here, so the rectangle can be arbitrarily long. See also Figure 3. Note that connectedness of the graph forces squares to have the same side length in any hexagonal realization.

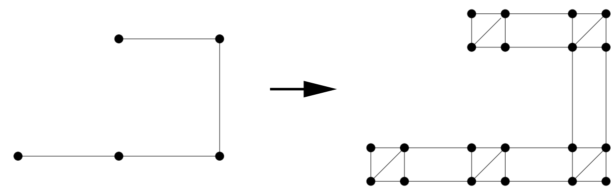

Figure 3: Replacing each vertex by a square, and each edge by a rectangle.

With this reduction, any realization of the representation orthogonal is easily converted into a realization of the corresponding hexagonal representation. To enforce edge length constraints, we modify the reduction further. For each edge, we thus far used a rectangle. We now add a *belt* by subdividing the two edges of the rectangle that are incident to vertices, and inserting a sequence of $2K+1$ squares with diagonals. (Recall that K is the upper bound on the maximum edge length.) See Figure 4.

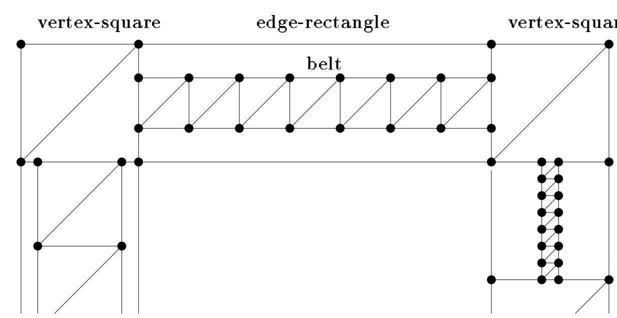

Figure 4: Adding belts to edge-rectangles.

Note that while all vertex-squares still must have the same height and width, the size of the belt is quite flexible: its height (for a horizontal edge) can be as little as one unit (but then the edge has to be fairly short), or it can be almost the height of the vertex-squares (but then the edge has to be very long.)

Lemma 5 *The orthogonal representation has a realization (on the rectangular grid) with maximum edge length at most K if and only if the hexagonal representation has a realization (on the hexagonal grid).*

Proof. Assume first we have a realization of the orthogonal representation with maximum edge length K. Scale the drawing by a factor of $\ell = 4K + 2$, i.e., make all edges ℓ times as long. Then replace each vertex with a square of side length $2K + 1$ centered at that vertex. Replace each edge with a rectangle connecting the

[1] Our reduction is loosely inspired by the reduction in [9], which converted a polygon into a hexagonal graph drawing, but proved NP-hardness of morphing such graph drawings instead.

squares at its endpoints. We want to place the belt inside this rectangle, but for this, need to show that if we scale the belt as to fit the edge length, then it is neither too small (it fits on a grid) nor too big (its height is less than the edge-rectangle's height.)

Consider one edge of the orthogonal drawing (we assume that it is horizontal for ease of description), and let its length be L, $1 \leq L \leq K$. After scaling the drawing this edge has length $\ell \cdot L$, and after inserting squares for vertices (which cuts off half a square at each end), the edge-rectangle has width

$$W := \ell \cdot L - (2K+1) = (4K+2)L - (2K+1).$$

Note that W is also the width of the belt for this edge. Since the belt has $2K+1$ squares, the height of the belt hence is $2L-1$. By $L \geq 1$, the height of the belt is $2L-1 \geq 1$, so the edges of the belt have length at least 1. Also, $2L-1$ is an integer, so the belt itself can be drawn on the grid. Finally by $L \leq K$, the height of the belt is $2L-1 \leq 2K-1 = (2K+1)-2$, so the height of the belt leaves two units space at the sides of the vertex-squares, and the belt hence can be placed inside the edge-rectangle on grid points.

For the other direction, given the hexagonal drawing, we can place each vertex at the center of the corresponding square and route the edge correspondingly. Clearly this gives an orthogonal drawing. With similar computations as above one can show that after scaling it down by a factor of $4K+2$, the maximum edge length is at most K and the drawing can be realized on a grid. $\square$

Theorem 6 *Testing whether a hexagonal representation can be realized in a grid drawing is NP-hard.*

We note here that "grid drawing" is not an essential part of the NP-hardness. As long as edges have to be drawn with non-zero length, the NP-hardness proof (with a slightly different scale factor) still holds.

Our NP-hardness result (and in fact, all of this paper) was about planar graphs with planar representations. However, the idea of converting an orthogonal representation into a hexagonal representation works similarly even if the graph is not planar; one can hence prove NP-hardness of realizations of (similarly defined) nonplanar hexagonal drawings, using e.g. the NP-hardness of minimizing the area of orthogonal drawings [4].

6 Conclusion

For hexagonal representations, the case is mostly closed: There is not always a drawing for a representation; even if there is one, it may have exponential edge length; and testing whether there is one is NP-hard. k-gonal drawings for $k \geq 4$ cannot be any easier than hexagonal drawings.

One remaining open problem is to ask similar questions in 3D. Here testing whether a realization exists is NP-hard even for orthogonal realizations [8]. Special cases that can be solved are paths [1, 6] and 3-connected planar graphs that are graphs of convex polyhedra [3]. Another special case that we are interested in is polyhedra. Thus, given a graph with orthogonal edge directions in 3D, is there an orthogonal polyhedron for which this is the graph? Is there an orthgonally convex polyhedron for which this is the graph? It is easy to see that the answer is sometimes "no", but is it NP-hard to test whether there is one? (The proof of [8] does not cover this case.)

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