Conflict-Free Coloring of Points on a Line with respect to a Set of Intervals

Matthew J. Katz* Nissan Lev-Tov* Gila Morgenstern*

Abstract

We present a 2-approximation algorithm for CF-coloring of points on a line with respect to a given set of intervals. The running time of the algorithm is $O(n \log n)$.

1 Introduction

The model of *Conflict-Free Coloring* (CF-Coloring) was introduced in [6] and further studied in [1, 7, 8]. This model arises from frequency assignment problems in cellular networks. In such networks each base station is assigned a certain frequency and transmits data in this frequency within some given region. Clients can connect to each other only via base stations and are able to scan frequencies in search for a base station that is received well. The quality of the reception depends on the noise caused by signals transmitted by other base stations that reach the client. In this context, a base station is received well if all other base stations that reach the client are assigned other frequencies and thus cannot interfere.

In the original CF-coloring model, the network is required to serve clients at any location that is reached by some base station. Thus, one must assign frequencies to the base stations, such that every point in each transmission region is supplied. The problem objective is to minimize the number of frequencies assigned.

Note that the CF-coloring problem is essentially different from the regular vertex coloring problem in the corresponding geometric intersection graph, as the latter completely forbids using the same color for intersecting regions. However, the NP-completeness proof of [5] for minimum coloring of unit disks, can be adapted to the case of CF-coloring of unit disks (that is, where each transmission region is a unit disk). Since this proof uses a reduction from coloring planar graphs, it also follows that CF-coloring of unit disks is hard to approximate within ratio $4/3 - \epsilon$, for any $\epsilon > 0$.

Algorithms that use $O(\log n)$ colors (where n is the number of regions) are given in [6] for CF-coloring of disks, axis-parallel rectangles, regular hexagons, and general congruent centrally symmetric convex regions in the plane. It is shown there that for a certain arrangement of the regions, called a "chain", $\Omega(\log n)$ colors are needed in order to supply all the points within the coverage area.

A dual model, in which points are colored with respect to regions was also defined in [6] and further studied in [8]. In this paper we consider a special one dimensional version of this problem. We are given a set of intervals on the real line, where the left and right endpoints of each interval belong to the set $\{1, 2, \ldots, n\}$. One must assign a color to each of the points $\{1, 2, \ldots, n\}$, such that all the given intervals are supplied. That is, for each interval there exists a point, among the points lying in the interval, whose color is unique. This problem is equivalent to the problem of CF-coloring a chain of unit disks, where it is necessary to supply only a given subset of the cells of the arrangement of the disks. This idea is actually a natural extension of the original CF-coloring model, since in many applications good reception is needed only in some locations.

Recently, online versions of CF-coloring of points with respect to intervals have been introduced [2, 3]. In these versions one must assign colors to points that arrive online, in order to supply all the intervals. Note that our model is different since only a given subset of the intervals must be supplied.

In Section 3 we consider two special cases of CF-coloring and present algorithms for these cases, achieving CF-coloring with only two (non-zero) colors. In Section 4 we present a 2-approximation algorithm for CF-coloring of points with respect to a given set of intervals. Note that it is still not clear whether this problem is polynomial or not. In Section 5, we analyze the complexity of the algorithms presented in this paper.

2 Preliminaries

Let $\mathcal{P} = \{p_1, p_2, ..., p_m\}$ be a set of points on the line, and let $\mathcal{R} = \{I_1, I_2, ..., I_n\}$ be a set of intervals. A *CF-coloring of* $\mathcal{P}$ *with respect to* $\mathcal{R}$ is a function $\chi : \mathcal{P} \longrightarrow \mathbb{N}$, such that for each $I \in \mathcal{R}$, at least one of the points of $\mathcal{P}$ that lie in I has a unique color; we say that such a point p *supports* I and also that its color $\chi(p)$ *supports* I. In this paper we use 0 as a null color, that cannot support any interval. A point whose color is 0 is considered colorless. This is also the default color, so that the

*Department of Computer Science, Ben-Gurion University, Israel; {matya, nissanl, gilamor}@cs.bgu.ac.il. N. Lev-Tov and G. Morgenstern were partially supported by the Lynn and William Frankel Center for Computer Sciences.

color of a point whose color was not fixed explicitly is considered to be 0.

By the definition of CF-coloring, we must require that for each interval $I \in \mathcal{R}$ there is a point of $\mathcal{P}$ that lies in I. Also, a point of $\mathcal{P}$ that does not lie in any interval of $\mathcal{R}$ is irrelevant, and can be colored 0. Thus, we assume that each interval of $\mathcal{R}$ covers at least one point of $\mathcal{P}$, and each point of $\mathcal{P}$ lies in at least one interval of $\mathcal{R}$.

For each $I \in \mathcal{R}$, we denote by $l(I)$ and $r(I)$ the left and right endpoints of I, respectively. We denote by $\vdash_I$ (resp. $\dashv_I$) the subset of intervals whose left (resp. right) endpoint is in I (including I itself). That is $\vdash_I = \{I' : l(I') \in I\}$ and $\dashv_I = \{I' : r(I') \in I\}$.

Let $I \in \mathcal{R}$ and let $p_l, p_r \in \mathcal{P}$ be the leftmost and rightmost points of $\mathcal{P}$ that lie in I. The interval $[p_l, p_r]$ is obtained from I by removing from both its ends pieces that are empty of points of $\mathcal{P}$. Obviously, for each point $p \in \mathcal{P}$ it holds that $p \in I$ if and only if $p \in [p_l, p_r]$, and hence we may assume that all intervals $I \in \mathcal{R}$ have their endpoints in $\mathcal{P}$.

Finally, for a subset $\mathcal{R}'$ of intervals, we denote by $Range(\mathcal{R}')$ the range defined by the intervals of $\mathcal{R}'$; that is $Range(\mathcal{R}') = [\min_{I \in \mathcal{R}'}\{l(I)\}, \max_{I \in \mathcal{R}'}\{r(I)\}]$.

3 Two Special Cases

In this section we consider two special cases of our CF-coloring problem, in which the set of intervals $\mathcal{R}$ satisfies some additional property. In both cases we obtain a CF-coloring using only two non-zero colors.

CF-coloring of points w.r.t. a set of non-nested intervals. Let $\mathcal{P}$ and $\mathcal{R}$ be as above. In addition, assume that there are no two intervals $I_1, I_2 \in \mathcal{R}$ such that $I_1 \subset I_2$. Algorithm NNCFCP (Non-Nested CF-Coloring of Points) below, outputs a CF-coloring χ of $\mathcal{P}$ with respect to $\mathcal{R}$ using only two non-zero colors.

Algorithm 1 NNCFCP

$IS \leftarrow$ maximum independent set of $\mathcal{R}$
for each $I \in IS$ **do**
  $\chi(l(I)) \leftarrow 1$, $\chi(r(I)) \leftarrow 2$
end for

Clearly, each $I \in IS$ is supported (twice). Assume there is an interval $I \notin IS$ that is not supported. If neither 1 nor 2 occur in I, then, since I is not contained in any other interval of $\mathcal{R}$, we have that $IS \cup \{I\}$ is independent, contradicting the fact that IS is maximum independent. Otherwise, both 1 and 2 occur in I at least twice each, implying that there exists an interval $I' \in IS$ such that $I' \subset I$ — a contradiction. Thus we conclude that χ is a CF-coloring and obtain the following lemma.

Lemma 1 *Let $\mathcal{P}$ be a set of points and let $\mathcal{R}$ be a set of intervals. If there are no two intervals $I_1, I_2 \in \mathcal{R}$ such that $I_1 \subset I_2$, then there exists a CF-coloring of $\mathcal{P}$ with respect to $\mathcal{R}$ that uses only two non-zero colors.*

CF-coloring of unit intervals w.r.t. a set of points. Let $\mathcal{R}$ be a set of unit intervals, and let $\mathcal{P}$ be a set of points such that $\mathcal{P}$ is covered by the intervals of $\mathcal{R}$. Algorithm UICFCI (Unit Intervals CF-Coloring of Intervals) below outputs a CF-coloring χ of $\mathcal{R}$ with respect to $\mathcal{P}$ using only two non-zero colors. (In this case the endpoints of the intervals of $\mathcal{R}$ are not necessarily points of $\mathcal{P}$.)

Algorithm 2 UICFCI

$IS \leftarrow$ maximum independent set of $\mathcal{R}$
$A \leftarrow \emptyset$
$last \leftarrow \emptyset$
for each $I \in IS$ by increasing right endpoint **do**
  if $I \cap last = \emptyset$ **then**
    add to A the leftmost interval in $\dashv_I$
  end if
  add to A the rightmost interval in $\vdash_I$
  $last \leftarrow$ the rightmost interval in $\vdash_I$
end for
$color \leftarrow 1$
for each $I \in IS \cup A$ by increasing right endpoint **do**
  $\chi(I) \leftarrow color$
  $color \leftarrow (color \mod 2) + 1$
end for

Note that $\mathcal{P}$ is covered by the intervals of $IS \cup A$. Note also that each point of $\mathcal{P}$ lies in at most two intervals of A and at most one interval of IS; that is, in at most three colored intervals. Now let p be any point in $\mathcal{P}$. The intervals of $IS \cup A$ that cover p appear one after the other in the final coloring loop, hence at least one of them is colored by a unique color. This interval supports p. Thus we conclude that χ is a CF-coloring and obtain the following lemma.

Lemma 2 *Let $\mathcal{R}$ be a set of unit intervals and let $\mathcal{P}$ be a set of points, then there exists a CF-coloring of $\mathcal{R}$ with respect to $\mathcal{P}$ that uses only two non-zero colors.*

Remark. It is not surprising that in the latter case it is also possible to manage with only two non-zero colors, since it is almost dual to the former case.

4 CF-Coloring of Points w.r.t. a Set of Intervals

Let $\mathcal{P}$ be a set of points and let $\mathcal{R}$ be a set of intervals (as discussed in Section 2). We assume that no two intervals have a common right endpoint (see remark at the end of this section). Algorithm CFCP (CF-Coloring of Points) below, outputs a CF-coloring of $\mathcal{P}$ with respect to $\mathcal{R}$.

Algorithm 3 CFCP

for each $I \in \mathcal{R}$ by increasing right endpoint **do**
  if I is already supported by some point to the left of $r(I)$ **then**
    $\chi(r(I)) \leftarrow 0$
  else
    $\chi(r(I)) \leftarrow$ smallest (non-zero) color that does not yet appear in I
  end if
end for

Note that the algorithm assigns colors only to points that are a right endpoint of some interval. We show below that despite this rather severe restriction, the algorithm computes a 2-approximation.

Observation 1 *For each $I \in \mathcal{R}$, if $\chi(r(I)) \neq 0$, then $\chi(r(I))$ supports I (i.e. occurs in I only once) and there is no other color that supports I.*

Observation 2 *For each $I \in \mathcal{R}$, if $\chi(r(I)) \neq 0$, then all colors smaller than $\chi(r(I))$ occur in I at least twice each.*

Notice that if $\chi(r(I)) = 0$, then it is possible that I is supported by several colors. Let $\mu(I)$ be the maximum among all colors supporting I.

Lemma 3 *Let $I \in \mathcal{R}$. Then $\mu(I)$ is the maximum among* all *colors occurring in I.*

Proof. Assume there is a color $\alpha > \mu(I)$ that occurs in I. By definition, α does not support I, i.e., it occurs in I at least twice at $r(I_1)$ and at $r(I_2)$. Assume, e.g., that $r(I_1) < r(I_2)$. By Observation 1, $\alpha = \mu(I_2)$, and since also $\alpha = \chi(r(I_1))$, we have that $r(I_1) \notin I_2$, thus I_2 is fully contained in I.

But, since $\alpha > \mu(I)$, Observation 2 implies that the color $\mu(I)$ occurs at least twice in I_2 and hence in I, contradicting the uniqueness of $\mu(I)$ in I. □

In the next lemma we bound from below the number of colors used by an optimal coloring algorithm, assuming, e.g., that all points lie in one large interval I and the rest of the intervals can be partitioned into two subsets with disjoint ranges.

Lemma 4 *Assume $\mathcal{R}_1, \mathcal{R}_2 \subseteq \mathcal{R}$ and $I \in \mathcal{R}$, such that (i) Range($\mathcal{R}_1$) $\cap$ Range($\mathcal{R}_2$) $= \emptyset$, and (ii) Range($\mathcal{R}_1 \cup \mathcal{R}_2$) $\subseteq I$. Put $\mathcal{R}' = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \{I\}$, and denote by $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) the subset of points of $\mathcal{P}$ that lie in Range($\mathcal{R}_1$) (resp. Range($\mathcal{R}_2$)). Finally, let χ_1 (resp. χ_2) be an optimal CF-coloring of $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) with respect to $\mathcal{R}_1$ (resp. $\mathcal{R}_2$), and let χ be an optimal CF-coloring of $\mathcal{P}$ with respect to $\mathcal{R}'$. Then $|\chi| > \min\{|\chi_1|, |\chi_2|\}$, where $|\chi|$ is the number of colors used by χ.*

Proof. Assume first that $|\chi_1| > |\chi_2|$. Clearly, any CF-coloring of $\mathcal{P}$ with respect to $\mathcal{R}'$ requires at least $|\chi_1|$ colors, thus $|\chi| \geq |\chi_1| > |\chi_2| = \min\{|\chi_1|, |\chi_2|\}$.

Assume now that $|\chi_1| = |\chi_2|$. Again, any CF-coloring of $\mathcal{P}$ with respect to $\mathcal{R}'$ requires at least $|\chi_1|$ colors. Moreover, the coloring χ defined below is a CF-coloring of $\mathcal{P}$ with respect to $\mathcal{R}'$ using $|\chi| = |\chi_1| + 1 > \min\{|\chi_1|, |\chi_2|\}$ colors. Let q be any point of $\mathcal{P}$ that lies in I, then χ is defined as follows:

$$\chi(p) = \begin{cases} |\chi_1| + 1 & \textit{if } p = q \\ \chi_1(p) & \textit{if } p \in \mathcal{P}_1 \\ \chi_2(p) & \textit{if } p \in \mathcal{P}_2 \\ 0 & \textit{otherwise} \end{cases}$$

We claim that χ is optimal. Assume there is a CF-coloring χ' of $\mathcal{P}$ with respect to $\mathcal{R}'$ that uses (beside maybe 0) only the colors $\{1, \ldots, |\chi_1|\}$, and let χ'_1 (resp. χ'_2) be the coloring of $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) induced by χ'. Let α be the color (in χ') of a point supporting I, then at least one of the colorings χ'_1, χ'_2 does not use the color α. Assuming, e.g., that χ'_1 does not use α, we get that χ'_1 is a CF-coloring of $\mathcal{P}_1$ with respect to $\mathcal{R}_1$ that uses less than $|\chi_1|$ colors, contradicting the optimality of χ_1. □

At this point, we note that (i) removing from $\mathcal{R}$ all intervals I with $\chi(r(I)) = 0$ and re-coloring $\mathcal{P}$ with respect to the remaining intervals (using CFCP again), results in exactly the same coloring, and (ii) the removal of intervals from $\mathcal{R}$ can only decrease the number of colors required by an optimal CF-coloring. Therefore, when determining the approximation ratio of CFCP, we may assume that all such intervals have been removed.

Lemma 5 *For each interval $I \in \mathcal{R}$, if $\chi(r(I)) = k$ (i.e., if CFCP assigned the color k to $r(I)$), then any CF-coloring of $\mathcal{P} \cap I$ with respect to $\dashv_I$ uses at least $\lceil \frac{k}{2} \rceil$ colors.*

Proof. By induction on k. The claim is clearly true for $k \leq 2$. Let $k > 2$ and assume the claim is true for any $t \leq k - 1$. We prove that it is also true for k.

Let I be an interval with $\chi(r(I)) = k$. Then by Observation 2 the color $k-1$ occurs in I (at least) twice, at $r(I'_1)$ and at $r(I'_2)$. Assume, e.g., that $r(I'_1) < r(I'_2)$. Since $r(I'_2) = k - 1 \neq 0$, we have (by Observation 1) that $k - 1$ supports I'_2, hence $r(I'_1) \notin I'_2$, implying that I'_2 is fully contained in I (see Figure 1(a)).

Now, the color $k - 2$ occurs (at least) twice in I'_2, at $r(I''_1)$ and at $r(I''_2)$, and assume $r(I''_1) < r(I''_2)$. Since (by Lemma 3) $k - 2$ is the maximum among all colors occurring in I''_1, $r(I'_1)$ cannot lie in I''_1, implying that both I''_1 and I''_2 are fully contained in I.

Recall that we are assuming that all intervals have a non-zero color at their right endpoint. Therefore by applying Observation 2 and Lemma 3 we obtain that $r(I''_1)$

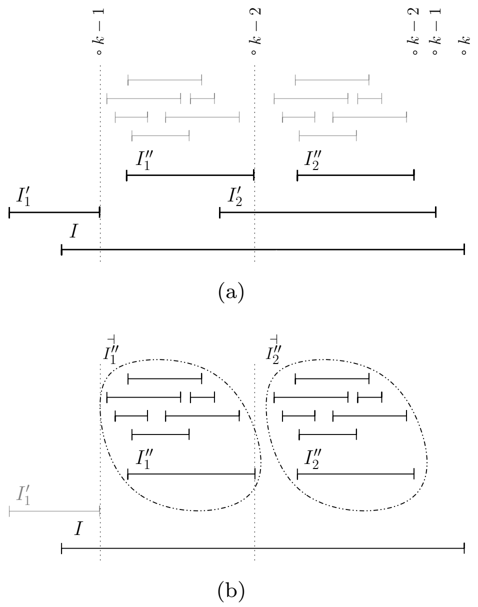

Figure 1: (a) Intervals I, I_1', I_2', I_1'', and I_2''. (b) The ranges of $\vec{I_1''}$ and $\vec{I_2''}$ are disjoint and contained in I.

does not lie in any of the intervals of $\vec{I_2''}$ and that $r(I_1')$ does not lie in any of the intervals of $\vec{I_1''}$. Thus, as Figure 1(b) illustrates, the set $\vec{I_1''} \cup \vec{I_2''} \cup \{I\}$ satisfies the conditions of Lemma 4. By the induction hypothesis, any optimal CF-coloring of the points of $\mathcal{P} \cap Range(\vec{I_1''})$ with respect to $\vec{I_1''}$ uses at least $\lceil \frac{k-2}{2} \rceil$ colors, and similarly for I_2''. Thus, by Lemma 4 we obtain that any optimal coloring of $\mathcal{P}$ with respect to the set $\vec{I_1''} \cup \vec{I_2''} \cup \{I\}$ uses at least $\lceil \frac{k-2}{2} \rceil + 1 = \lceil \frac{k}{2} \rceil$ colors. □

We can now state the main theorem.

Theorem 6 *CFCp computes a 2-Approximation.*

Proof. Let χ be the coloring produced by CFCp, and put $|\chi| = k$. Let $I \in \mathcal{R}$ be the first interval for which the color k was used by CFCp. By Lemma 5, any optimal CF-coloring of the points of $\mathcal{P} \cap I$ with respect to $\vec{I}$ uses at least $\lceil \frac{k}{2} \rceil$ colors. Let OPT be an optimal CF-coloring of $\mathcal{P}$ with respect to $\mathcal{R}$. Obviously OPT uses at least this number of colors, thus we conclude that $|\chi| \leq 2|OPT|$. □

Remark. Our proof does not apply to the version where intervals may share endpoints. In the full paper we present a 4-approximation for this version.

5 Complexity

In this section we examine the complexity of the algorithms mentioned above. Both algorithms presented in Section 3 first find a maximum independent set *IS* of $\mathcal{R}$. This can be done in time $O(n \log n)$. Subsequently, Algorithm NNCFCp only colors the endpoints of the intervals of *IS*. Concerning Algorithm UICFCi, it considers the intervals of *IS* one after the other, and for each such interval I it finds the leftmost (resp. rightmost) interval in $\mathcal{R}$ with endpoint in I. Since $\mathcal{R}$ is a set of unit intervals, this can all be done in time $O(n \log n)$. Thus both algorithms of Section 3 run in time $O(n \log n)$.

Algorithm CFCp considers the intervals in $\mathcal{R}$, one by one, by increasing right endpoint. For each such interval I one must find (based on Lemma 3) the maximal color α among the colors already in I, and determine whether α is unique in I. If it is unique, then I's right endpoint is colored 0, otherwise, it is colored $\alpha + 1$. In order to support such queries one can employ a semi-dynamic range tree (insertions only) [4], and maintain some additional information in its nodes. Thus the total running time of Algorithm CFCp is also $O(n \log n)$.

References

[1] N. Alon, S. Smorodinsky. Conflict-free colorings of shallow discs. In *Proc. 22nd ACM Sympos. on Comput. Geom.*, pp. 41–43, 2006.

[2] A. Bar-Noy, P. Cheilaris, S. Smorodinsky. Conflict-free coloring for intervals: from offline to online. In *Proc. 18th ACM Sympos. on Parallelism in Algorithms and Architectures*, pp. 128–137, 2006.

[3] K. Chen, A. Fiat, H. Kaplan, M, Levy, J. Matoušek, E. Mossel, J. Pach, M. Sharir, S. Smorodinsky, U. Wagner, E. Welzl. Online conflict-free coloring for intervals. *SIAM J. Comput.* 36:1342–1359, 2006.

[4] Y.-J. Chiang, R. Tamassia. Dynamic algorithms in computational geometry. *Technical Report* CS-91-24, Dept. of Computer Science, Brown Univ., 1991.

[5] B.N. Clark, C.J. Colbourn, D.S. Johnson. Unit disk graphs. *Discrete Math.* 86:165-177, 1990.

[6] G. Even, Z. Lotker, D. Ron, S. Smorodinsky. Conflict-free colorings of simple geometric regions with applications to frequency assignment in cellular networks. *SIAM J. Comput.* 33(1):94–136, 2003.

[7] S. Har-Peled, S. Smorodinsky. On conflict-free coloring of points and simple regions in the plane. *Discrete & Comput. Geom.* 34:47–70, 2005.

[8] J. Pach, G. Tóth. Conflict free colorings. *Algorithms & Combinatorics* 25:665-672, 2003.