A Lower Bound on the Area of a 3-Coloured Disc Packing

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Abstract

Given a set of unit-discs in the plane with union area A, what fraction of A can be covered by selecting a pairwise disjoint subset of the discs? Rado conjectured 1/4 and proved 1/4.41. Motivated by the problem of channel-assignment for wireless access points, we consider a variant where the selected subset of discs must be 3-colourable, with discs of the same colour pairwise-disjoint. For this variant of the problem, we conjecture that it is always possible to cover at least 1/1.41 of the union area and prove 1/2.77. We also provide an $O(n^2)$ algorithm to select a subset achieving our 1/2.77 bound, and a proof that this problem is 3SUM-hard, providing strong evidence that our algorithm is optimal.

1 Introduction

R. Rado studied the following problem: What is the largest c_1 such that, given any arrangement of unit-discs D in the plane, we can always select a pairwise disjoint subset of discs that cover at least a fraction c_1 of the area of the union of D? The best we can hope for is $c_1 = \frac{1}{4}$, corresponding to the case shown in Figure 1 where a large number of unit-discs share a very small intersection—the common intersection prevents us from selecting more than a single disc, while the union area of all discs approaches 4π. Rado proved a lower bound of $c_1 = \frac{\pi}{8\sqrt{3}} \approx \frac{1}{4.41}$ [11] and conjectured the lower bound $c_1 = \frac{1}{4}$.

In this paper, we study a variant of the above problem in which the disjointness constraint on the selected subset of discs is relaxed slightly. Given an arrangement of unit-discs D in the plane, we want to find the largest c_3 such that we can always select and 3-colour a subset of the discs C such that their union area covers at least a fraction c_3 of the union area of D, under the constraint that same-coloured discs must be pairwise disjoint. We will prove that, for any given arrangement of unit-discs, we can always achieve greater than $c_3 \approx \frac{1}{2.77}$. Our result is stated formally in Theorem 1 below.

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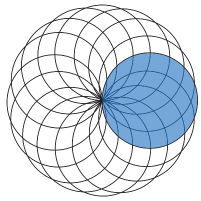

Figure 1: Given a set of circles arranged in a circle with a very small mutual intersection, the most area we can cover is $\frac{\pi}{4\pi} = \frac{1}{4}$.

Note that it is not possible to achieve better than $c_3 \approx \frac{1}{1.41}$, corresponding to selecting three minimally pairwise intersecting discs in the arrangement shown in Figure 1. We conjecture that this bound is achievable for any arrangement of unit-discs.

Theorem 1 *Let D be a collection of unit-discs in the plane with union area A. For C a 3-coloured subset of D with same-coloured discs pairwise disjoint, let A_C denote C's union area. We can always select a C such that $c_3 = \frac{A_C}{A} \geqslant \frac{1}{2.77}$.*

The rest of this paper is organized as follows. In Section 2 we present our motivation for exploring this problem and discuss some related problems. In Section 3 we review Rado's proof and discuss some techniques that will make also make an appearance in our proof. In Sections 4 and 5 we present our proof of Theorem 1 and a supporting lemma. In Section 6 we present an $O(n^2)$ time algorithm to select a subset which achieves our proven bound, and prove that this problem is 3SUM-hard. Finally, Section 7 discusses bounds for k-colours.

2 Motivation and Related Work

Wi-Fi (IEEE 802.11) wireless networks are becoming a ubiquitous feature in modern businesses, universities, parks, etc. In a typical Wi-Fi deployment scenario, a set of candidate locations are determined for wireless access points (APs). A subset of the candidate locations must be chosen along with a channel assignment for each installed AP in order to maximize the area covered by the wireless network while minimizing interference. Interference occurs when two APs using the same channel are within range of one another, preventing users in range of both APs from communicating with either AP.

Supposing three channels are available, the wireless network deployment problem corresponds to the 3-coloured disc packing problem discussed in Section 1. Each disc represents the coverage area of a potential AP placed at the disc's center, and a 3-coloured disc packing represents a deployment of APs with channel assignments. Moreover, a study by wireless hardware maker Cisco recommends that wireless network deployments only use three channels for networks with a high volume of users [6].

A more sophisticated formalization of the deployment problem allows discs assigned to the same channel to overlap but only counts the area where there is no interference—i.e. the area of the set of points covered by only one disc on some channel. In terms of colouring, the problem is to colour a subset of the given discs to maximize the area of $\{p \in \mathbb{R}^2 \mid$ for some colour, point p is in exactly one disc of that colour$\}$. We call this the *1-covered area*.

Asano et al. [3] proved that it is always possible to achieve approximately $\frac{A}{4.37}$ 1-covered area using only one colour. This model has also been considered with respect to two other optimization problems. For the problem of maximizing the 1-covered area using one colour, there is a PTAS due to Chen et al. [5]. Note that unlike our worst-case area bound, they compare the area covered to the optimum *for that instance*.

Another well-explored problem is *conflict-free colouring*—here the goal is to minimize the number of colours needed to 1-cover the whole area, i.e. the union of the given discs. Even et al. [7] prove that $O(\log n)$ colours are always sufficient and sometimes necessary for any given discs of general radii. There is also work on online algorithms for conflict-free colouring [8], and on conflict-free colouring of regions other than discs [10].

3 Preliminaries

The basic idea behind our proof is similar to that used by Rado [11]. His idea was to impose a regular triangular lattice of side length 4 over the set of discs D and, for each point in the union of D, to select one disc containing that lattice point. The side length of the lattice guarantees that discs selected in this manner will be pairwise disjoint (see Figure 2). Thus, supposing we can prove a lower bound of k on the number of lattice points which intersect any given set of discs, we immediately obtain a lower bound of $c_1 = \pi k$ for Rado's version of the problem. Such a lower bound on k is given by Rado in [11], proving that given a set of geometric objects in the plane with union area A, and a triangular lattice in which each triangle has area α, we can always find a translation of the lattice such that it intersects $\frac{A}{2\alpha}$ points in the union area. A straightforward subsitution of $\alpha = 4\sqrt{3}$ (the area of an equilateral triangle with side length 4) yields Rado's result of $c_1 = \frac{\pi}{8\sqrt{3}}$.

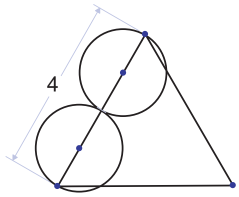

Figure 2: A triangular lattice ensures that selected discs are pairwise disjoint.

4 Proof of Theorem 1

Proof. To solve our variation of the problem we use a finer triangular lattice with side length $\frac{4\sqrt{3}}{3}$. The points of the lattice are 3-coloured such that no two lattice points of the same colour are adjacent. For any placement of the lattice, select a subset C of D as follows: for each lattice point p in the union of D, select a disc containing p and assign the disc the colour of p. The side length of the lattice ensures that no disc contains two lattice points so the selection and colouring are well-defined. It also ensures that discs assigned the same colour are pairwise disjoint (see Figure 3).

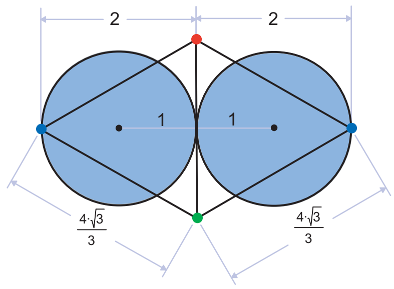

Figure 3: A finer 3-coloured lattice ensures only that same-coloured selected discs are pairwise disjoint.

Also observe that, by the result of Rado in [11], we can position the lattice to intersect the union area of D in at least $\frac{A\sqrt{3}}{8}$ points, so $|C| \geqslant \frac{A\sqrt{3}}{8}$. While same-coloured discs in C are pairwise disjoint, differently coloured discs may not be, so $|C|\pi$ is only an upper bound on A_C.

To derive a lower bound we will partition the union of C using the lattice's Voronoi tesselation which has regular hexagonal cells of side length $\frac{4}{3}$ (see Figure 4). Suppose disc $d \in C$ contains lattice point p which lies in hexagonal cell h. If we count only the area of $d \cap h$, and sum over all d, this gives a lower bound on A_C. Thus if we establish a lower bound Δ on the minimum possible area of $d \cap h$ then $A_C \geqslant |C|\Delta \geqslant \frac{A\sqrt{3}}{8}\Delta$. In Lemma 2, which we will prove in Section 5, we show that $\Delta \approx 1.6645$. From the lower bound on A_C we reach our intended lower bound on c_3 of

$$c_3 = \frac{A_C}{A} = \frac{A\sqrt{3}}{8}\Delta \approx \frac{A}{2.77}$$

□

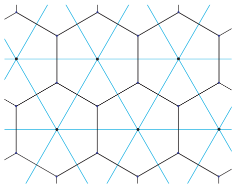

Figure 4: The Voronoi tessellation of the triangular lattice points forms a grid of regular hexagonal cells.

5 Minimum Disc Hexagon Intersection

Lemma 2 *Given a regular hexagon h with center point X and side length $\frac{4}{3}$, and any unit-disc d intersecting point X, the minimum area of intersection Δ between h and d is approximately* 1.6645*, or more precisely*

$$\Delta = \frac{\sqrt{3}}{36} + \frac{\sqrt{11}}{12} + \frac{\pi}{2} - \frac{1}{2}\arctan\left(\frac{5\sqrt{3}-\sqrt{11}}{5+\sqrt{11}\sqrt{3}}\right)$$

Proof. We use elementary geometry to argue that the minimum area of intersection is achieved by a disc d with X on its boundary. Then, parameterizing by the angle θ between the horizontal axis and the ray from X to the center of d, we use the symbolic geometry package Geometry Expressions to give a formula for the area of intersection and use Maple to compute 0's of the first derivative, finding that the minimum is as stated above, and occurs in the configuration shown in Figure 5. Further details are included in Appendix A. □

Our proof of Lemma 2 also shows that the lower bound $A_C \geq |C|\Delta$ is tight, as shown by the example in Figure 5 where the union of C is exactly partitioned by the hexagons and each disc intersects its hexagon in the minimum area Δ. However, note that this does not mean that our bound on c_3 is tight.

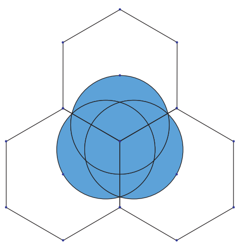

Figure 5: In this arrangement of selected discs, the lower bound on the contribution of each disc is realized.

6 Algorithm

In this section we give an $O(n^2)$ time algorithm to select and 3-colour a subset C of a set D of unit-discs so that the area bound given in Theorem 1 is realized. The proof of the theorem is constructive, and the only algorithmic issue is positioning the lattice so that at least $\frac{A\sqrt{3}}{8}$ lattice points are in $\cup D$ (the union of all discs in D). We give an $O(n^2)$ time algorithm for this lattice positioning problem. We also prove that the lattice positioning problem is 3SUM-hard, providing evidence that an $O(n^2)$ time algorithm is the best we can expect.

The idea, which comes from Rado's paper [11], uses the concept of the "fundamental cell" of a lattice—for this lattice the fundamental cell F consists of a pair of adjacent triangles (see Figure 3). Given an arbitrary placement of the lattice, each triangle of the lattice can be translated to F along with whatever part of $\cup D$ is contained in the triangle. The translated parts of $\cup D$ will overlap in F, and we want a point p of maximum depth k. Positioning the lattice with a lattice point at p ensures that k lattice points fall in $\cup D$. (This is how Rado obtains the bound $k \geqslant \frac{A}{2\alpha}$ where α is the area of one triangle.)

The one remaining detail is how to capture the translated portions of $\cup D$ so that we can compute a point of maximum depth. Our basic idea involves translating all the discs and then computing and traversing their arrangement. Each disc d intersects at most 4 translates of F. We make 4 translated copies of d, and record which translate of F they come from. Computing this set of translated discs D' takes $O(n)$ time. Computing the arrangement of D', $\mathcal{A}(D')$, takes $O(n^2)$ time using the incremental insertion algorithm of Chazelle and Lee [4].

It is easy to traverse $\mathcal{A}(D')$ to compute maximum depth in D'—the depth increases when we enter a disc and decreases when we exit. However, this is not quite what we want; we want depth with respect to $\cup D$ translated to F, which is different from depth in D' due to discs that overlap originally in D. Our solution is to traverse $\mathcal{A}(D')$ maintaining the depth c_i in each translate i of F. Note that there are $O(n)$ translates of F that intersect $\cup D$. We also maintain a count c of the number of non-zero c_i's. The maximum value of c over cells of $\mathcal{A}(D')$ gives us what we want.

We now prove that the lattice positioning problem discussed above is 3SUM-hard. A problem is *3SUM-hard* if it is harder than the problem of determining whether a set S of n integers contains three elements $a, b, c \in S$ such that $a + b + c = 0$. The best known algorithms for this problem take $O(n^2)$ and it is an open problem to do better [9].

Theorem 3 *The following problem is 3SUM-hard: Given an integer k, real number s, and a set D of n unit-discs in the plane, determine whether a triangular lattice of side length s can be positioned such that it intersects the union area of D in at least k points.*

Proof. We show that our problem is harder than the known 3SUM-hard problem of determining whether there is a point of depth k in a set of unit radius discs in the plane. The more general problem for variable radius discs is proved 3SUM-hard in [2] and the reduction is easily modified to produce unit radius discs.

Our reduction is as follows. Given a set D of unit radius discs in the plane, place an equilateral triangle T large enough to contain all of D. Expand T to a triangular lattice, and translate each disc of D to a different cell in the lattice. Let the translated set of discs be D'. Then there is a point of depth k in D iff the lattice can be translated to intersect D' in k points. This reduction takes linear time. □

7 Extension to k-Colours and Future Work

While it is fortuitous that our proof technique can handle 3 colours, since that is relevant for channel assignment in wireless networks, it is interesting to see what general bound can be derived for k-colours. Theorem 4 presents some preliminary results, demonstrating a bound for all k such that $k \in \{i^2 + ij + j^2 \mid i, j \in \mathbb{N}\}$. The number of such k lower than a given $x \in \mathbb{N}$ is given by $\Theta(\frac{x}{\sqrt{\log x}})$, so the set of such k is thin (density 0).

Theorem 4 *Given k colours, where $k \in \{i^2 + ij + j^2 \mid i, j \in \mathbb{N}\}$ we can select and colour a subset of discs such that same-coloured discs are disjoint and their union area covers at least $A\frac{1}{(1+\delta_k)^2}$ where $\delta_k = \frac{2}{\sqrt{3}}\left(\frac{2}{\sqrt{k}-\frac{2}{\sqrt{3}}}\right)$.*

Proof. For all $k \in \{i^2 + ij + j^2 \mid i, j \in \mathbb{N}\}$, distance $\sqrt{k}$ occurs in the unit triangular lattice L, and by $\frac{\pi}{3}$ rotational symmetry, an entire sublattice with side length $\sqrt{k}$ exists. Thus we can partition L into k triangular lattices of side length $\sqrt{k}$ and assign each a unique colour. We then scale L such that distance 2 separates the enclosing discs of Voronoi cells of same-coloured lattice points by applying a scaling factor of $\alpha_k = \frac{2}{\sqrt{k}-\frac{2}{\sqrt{3}}}$. Now, each disc in D is assigned to the Voronoi cell containing its center. We select from each Voronoi cell one associated disc (if there are any) and colour it to match the Voronoi cell's lattice point. Note that by our scaling, same-coloured selected discs cannot intersect.

If a point p is in $\cup D$ but is not in any selected disc, then the disc covering p intersects another disc with center in the same Voronoi cell, and the distance between their center points is less than the diameter of the Voronoi cell $\delta_k = \frac{2}{\sqrt{3}}\alpha_k$. Now, if all selected discs were blown up by a factor of $1 + \delta_k$, p would be covered by some selected disc and the union of selected discs would cover at most $A(1 + \delta_k)^2$. Thus, if we allow k colours, we can cover at least $A\frac{1}{(1+\delta_k)^2}$. □

We are also continuing work on the 3-colour version of the problem, and have recently improved our bound to approximately 1/2.09.

Acknowledgements

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A Proof of Lemma 2

Our first claim is that the minimum area of intersection is achieved by a disc d positioned such that X, the center point of hexagon h, lies on its boundary. Suppose this is not the case. By symmetry, it suffices to consider possible placements of Y, the center point of d, within the intersection of wedge BXK and the unit disc centered at X in Figure 6. For any position of Y, moving Y to the right along a line parallel to AB decreases the area of intersection, since the portion of $d-h$ lying above the supporting line of AB stays the same, and the portion of $d-h$ below the supporting line of AB strictly increases (by containment). Thus we can move Y to the right until it lies either on XK or the boundary of the disc centered at X. For Y on XK, moving Y towards K decreases $d \cap h$ because when the diameter of d parallel to BC lies strictly inside h, the area of $d-h$ increases (by containment), and when the diameter is not strictly contained in h, the area of $d \cap h$ decreases (by containment).

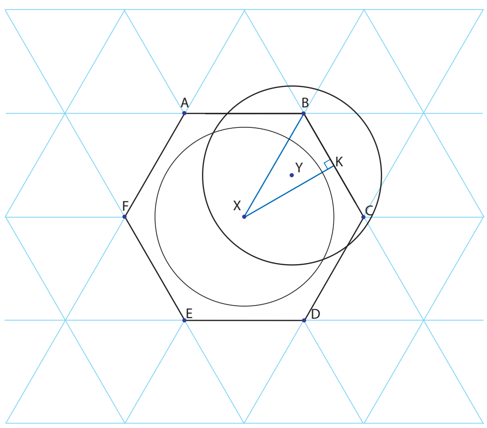

Figure 6: Sliding disc d outward from the center of h results in a smaller intersection between d and h

Thus we can restrict our attention to the minimum area of intersection with h among discs whose boundary contains point X. To find this minimum, we assume that $X = (0,0)$ and express the area of intersection $f(\theta)$ in terms of angle θ between the center of a disc d, the center X of h, and the x-axis. There are two general cases to consider. Case 1 occurs when d contains two vertices of h (e.g. Figure 7). Case 2 occurs when d only contains a single vertex of h (e.g. Figure 8). Note that in either case we can express the area of intersection as the sum of the area of a polygon and a circle sector. For instance, in Figure 7 the area of intersection is the sum of the area of polygon $ABCED$ and the area of the sector of d interior to angle ABC.

By symmetry, we need only consider the area of intersection for $0 \leqslant \theta \leqslant \frac{\pi}{6}$. As a result, we can use the cases shown in Figures 7 and 8 to derive a formula for

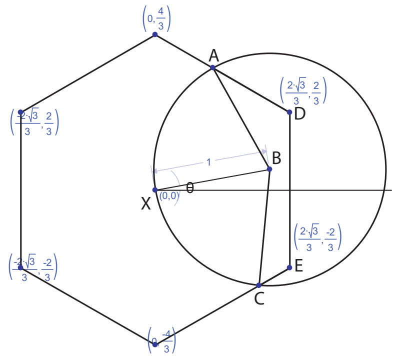

Figure 7: Calculating the intersection area as a function of angle θ (Case 1).

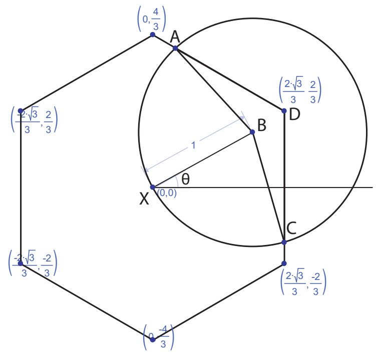

Figure 8: Calculating the intersection area as a function of angle θ (Case 2).

$f(\theta)$. Specifically, we use the symbolic geometry package Geometry Expressions to derive formulas relating θ and the intersection points between the boundaries of d and h (i.e. points A and C in Figures 7 and 8). The derived formulas for these points, along with formulas for the other points in Figures 7 and 8 are given in Appendix A.1 and A.2 respectively. From these formulas we express the area of intersection in terms of θ using standard formulas for the area of polygons and circle sectors. This gives us formula $f_1(\theta)$ for the area of intersection for $0 \leqslant \theta \leqslant \arccos(\frac{2}{3}) - \frac{\pi}{6}$ and formula $f_2(\theta)$ for the area of intersection for $\arccos(\frac{2}{3}) - \frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{6}$ (see Appendix A.1 and A.2. By symmetry, we extend this to formula $f(\theta)$ for $0 \leqslant \theta \leqslant \frac{\pi}{3}$ given in Appendix A.3.

A plot showing $f(\theta)$ for the interval $0 \leqslant \theta \leqslant \frac{\pi}{3}$ is given in Figure 9. Using Maple we find that $f'(\theta)$ (the first derivative of $f(\theta)$) is 0 for $\theta = \frac{\pi}{6}$. Thus, by symmetry, the minimum intersection occurs when X, the center point of d and a vertex of of h are collinear. Computing the value of $f(\theta)$ at any one of these points gives our value for Δ, specifically

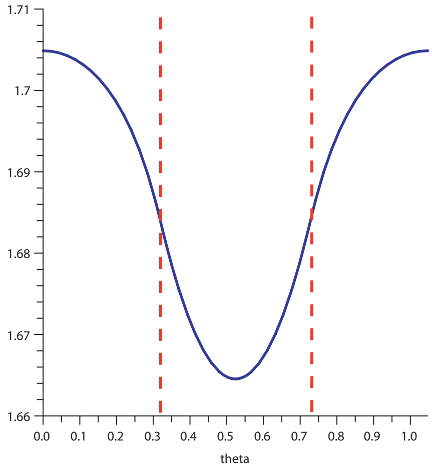

Figure 9: Plot showing the area of intersection for $0 \leqslant \theta \leqslant \frac{\pi}{3}$.

$$\begin{aligned}\Delta &= \frac{\sqrt{3}}{36} + \frac{\sqrt{11}}{12} + \frac{\pi}{2} - \frac{1}{2}\arctan\left(\frac{5\sqrt{3}-\sqrt{11}}{5+\sqrt{11}\sqrt{3}}\right) \\ &\approx 1.6645\end{aligned}$$

A.1 Derived Formula for $0 \leqslant \theta \leqslant \arccos(\frac{2}{3}) - \frac{\pi}{6}$ (see Fig. 7)

$$
\begin{aligned}
A_x(\theta) &= \frac{1}{3}\sqrt{3} - \frac{1}{2}\sqrt{1-\left(-\frac{2}{3}\sqrt{3}+\frac{1}{2}\sin(\theta)\sqrt{3}+\frac{1}{2}\cos(\theta)\right)^2}\sqrt{3} - \frac{1}{4}\sin(\theta)\sqrt{3} + \frac{3}{4}\cos(\theta)\\
A_y(\theta) &= 1 + \frac{1}{2}\sqrt{1-\left(-\frac{2}{3}\sqrt{3}+\frac{1}{2}\sin(\theta)\sqrt{3}+\frac{1}{2}\cos(\theta)\right)^2} + \frac{1}{4}\sin(\theta) - \frac{1}{4}\cos(\theta)\sqrt{3}\\
B_x(\theta) &= \cos(\theta)\\
B_y(\theta) &= \sin(\theta)\\
C_x(\theta) &= \frac{1}{3}\sqrt{3} - \frac{1}{2}\sqrt{1-\left(\frac{2}{3}\sqrt{3}+\frac{1}{2}\sin(\theta)\sqrt{3}-\frac{1}{2}\cos(\theta)\right)^2}\sqrt{3} + \frac{1}{4}\sin(\theta)\sqrt{3} + \frac{3}{4}\cos(\theta)\\
C_y(\theta) &= -1 - \frac{1}{2}\sqrt{1-\left(\frac{2}{3}\sqrt{3}+\frac{1}{2}\sin(\theta)\sqrt{3}-\frac{1}{2}\cos(\theta)\right)^2} + \frac{1}{4}\sin(\theta) + \frac{1}{4}\cos(\theta)\sqrt{3}\\
D_x(\theta) &= \frac{2}{3}\sqrt{3}\\
D_y(\theta) &= \frac{2}{3}\\
E_x(\theta) &= \frac{2}{3}\sqrt{3}\\
E_y(\theta) &= -\frac{2}{3}
\end{aligned}
$$

$$
\begin{aligned}
f_1(\theta) = {} & \frac{1}{2}\left(A_x(\theta)(B_y(\theta)-D_y(\theta)) + B_x(\theta)(C_y(\theta)-A_y(\theta)) + C_x(\theta)(E_y(\theta)-B_y(\theta)) + E_x(\theta)(D_y(\theta)-C_y(\theta)) + D_x(\theta)(A_y(\theta)-E_y(\theta))\right)\\
& + \frac{1}{2}\left(\pi + \arctan\left(\frac{-\left(-B_x(\theta)+C_x(\theta)\right)\left(-A_y(\theta)+B_y(\theta)\right)+\left(B_y(\theta)-C_y(\theta)\right)\left(A_x(\theta)-B_x(\theta)\right)}{\left(-B_x(\theta)+C_x(\theta)\right)\left(A_x(\theta)-B_x(\theta)\right)+\left(B_y(\theta)-C_y(\theta)\right)\left(-A_y(\theta)+B_y(\theta)\right)}\right)\right)
\end{aligned}
$$

A.2 Derived Formula for $\arccos(\frac{2}{3}) - \frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{6}$ (see Fig. 8)

$$
\begin{aligned}
A_x(\theta) &= \frac{1}{3}\sqrt{3} - \frac{1}{2}\sqrt{1-\left(-\frac{2}{3}\sqrt{3}+\frac{1}{2}\sin(\theta)\sqrt{3}+\frac{1}{2}\cos(\theta)\right)^2}\sqrt{3} - \frac{1}{4}\sin(\theta)\sqrt{3} + \frac{3}{4}\cos(\theta)\\
A_y(\theta) &= 1 + \frac{1}{2}\sqrt{1-\left(-\frac{2}{3}\sqrt{3}+\frac{1}{2}\sin(\theta)\sqrt{3}+\frac{1}{2}\cos(\theta)\right)^2} + \frac{1}{4}\sin(\theta) - \frac{1}{4}\cos(\theta)\sqrt{3}\\
B_x(\theta) &= \cos(\theta)\\
B_y(\theta) &= \sin(\theta)\\
C_x(\theta) &= \frac{2}{3}\sqrt{3}\\
C_y(\theta) &= -\frac{1}{3}\sqrt{-3+12\cos(\theta)\sqrt{3}-9\left(\cos(\theta)\right)^2} + \sin(\theta)\\
D_x(\theta) &= \frac{2}{3}\sqrt{3}\\
D_y(\theta) &= \frac{2}{3}
\end{aligned}
$$

$$
\begin{aligned}
f_2(\theta) = {} & \frac{1}{2}\left(A_x(\theta)(B_y(\theta)-D_y(\theta)) + B_x(\theta)(C_y(\theta)-A_y(\theta)) + C_x(\theta)(D_y(\theta)-B_y(\theta)) + D_x(\theta)(A_y(\theta)-C_y(\theta))\right)\\
& + \frac{1}{2}\left(\pi + \arctan\left(\frac{-\left(-B_x(\theta)+C_x(\theta)\right)\left(-A_y(\theta)+B_y(\theta)\right)+\left(B_y(\theta)-C_y(\theta)\right)\left(A_x(\theta)-B_x(\theta)\right)}{\left(-B_x(\theta)+C_x(\theta)\right)\left(A_x(\theta)-B_x(\theta)\right)+\left(B_y(\theta)-C_y(\theta)\right)\left(-A_y(\theta)+B_y(\theta)\right)}\right)\right)
\end{aligned}
$$

A.3 Derived Formula for $0 \leqslant \theta \leqslant \frac{\pi}{3}$

$$
f(\theta) = \begin{cases}
f_1(\theta) & \text{if } 0 \leqslant \theta < \arccos(\frac{2}{3}) - \frac{\pi}{6}\\
f_2(\theta) & \text{if } \arccos(\frac{2}{3}) - \frac{\pi}{6} \leqslant \theta < \frac{\pi}{6}\\
f_2(\frac{\pi}{3}-\theta) & \text{if } \frac{\pi}{6} \leqslant \theta < \arccos(\frac{2}{3})\\
f_1(\frac{\pi}{3}-\theta) & \text{if } \arccos(\frac{2}{3}) \leqslant \theta \leqslant \frac{\pi}{3}
\end{cases}
$$