Terminal Steiner Tree with Bounded Edge Length

Zhiyong Lin*

Abstract

In this paper, we study a Steiner tree related problem called "Terminal Steiner Tree with Bounded Edge Length": given a set of terminal points P in a plane, one is asked to find a Steiner tree T such that any point in P is a leaf in T and the length of each edge in T is no more than a constant b. The objective of the problem is to minimize the number of Steiner points in T. This problem is motivated from wireless network design and has various applications in wireless network area. We present a constant approximation algorithm for this problem in this paper.

1 Introduction

Consider a wireless sensor network with n sensors, each sensor has limited power so that it can only communicate with sensors within a limited range. Moreover, these sensors can not relay the messages from the neighbouring sensors due to its limited power and simple functionality. In order to make the network connected, we need to put some relays in this network. Since it has cost to install relays, we want to minimize the number of relays.

This application motivates the following problem. Given a set of terminal points P in a 2-dimension Euclidean space, one is asked to find a Steiner tree T such that any point in P is a leaf in T and the length of each edge in T is no more than a constant bound b. The objective of the problem is to minimize the number of Steiner points in T.

A practical variant of the well-known graph Steiner tree problem is investigated in [5, 3, 2, 8]. In this variant, every terminal point is required to be a leaf in the Steiner tree and the objective is to minimize the total length of the tree. In [5], the authors presented hardness results for this variant as well as a polynomial time approximation algorithm with performance ratio $\rho + 2$, where ρ is the best-known approximation ratio for the graph Steiner tree problem (currently $\rho \approx 1.550$, see [9]). The approximation ratio was improved to 2ρ in [3, 2], and further to $2\rho - \rho/(3\rho - 2)$ in [8].

Another variant of the Steiner tree problem called "Steiner tree with minimum number of Steiner points and bounded edge-length" is studied in [6, 7, 1]. The well known Steiner tree problem considers the minimum length of tree. While in this variant, the objective is to minimize the number of Steiner points under the constraint that the edge length of Steiner tree is bounded. [6] presented a simple polynomial time approximation algorithm whose worst-case performance ratio is 5. A careful analysis in [7] showed that the performance ratio of the algorithm in [6] is 4 and is tight. Moreover [1] showed that there exists a polynomial-time approximation with performance ratio 3 and there exists a polynomial-time approximation scheme under certain conditions.

Our problem is actually the combination of the above two variants of the Steiner tree problem. However algorithms for the above problems can not be applied to our problem. In this paper we present a polynomial time algorithm with constant performance ratio using the MST heuristic algorithm in [6].

2 Algorithm and Analysis

In this section, we present properties of the optimal solution. We also present our algorithm and show the quality of our algorithm.

Lemma 1 *There exists an optimal Steiner tree to the problem such that if we remove all terminal points and their corresponding edges, then for each Steiner point s and two edges su, sv, the angle $\angle usv$ is no less than $\pi/3$.*

Proof. Given an optimal Steiner tree T_{OPT}, we can modify it to satisfy the condition after removing all terminal points and corresponding edges. Consider any Steiner point s and two edges su and sv. Without loss of generality, we assume $|su| \geq |sv|$. If $\angle usv > \pi/3$, we have $|uv| < |su|$. Thus we can replace su with uv and get another optimal Steiner tree. By repeating this process, we will get an optimal Steiner tree such that for each Steiner point s and two edges su, sv, the angle $\angle usv$ is no less than $\pi/3$. □

2.1 Our algorithm

Our algorithm is based on the following intuitive observation. Intuitively, if some terminal points are close, we can treat them as one (i.e. use a steiner point to

*Department of Computer Science and Engineering, University at Buffalo, the State University of New York, Buffalo, NY 14260, USA, zlin@cse.buffalo.edu

cover them). So instead of building the Steiner tree on all terminal points, we group terminal points first and then build the Steiner tree on the groups. Thus we have the following algorithm $TSTBEL$ as in Algorithm 1.

Algorithm 1 TSTBEL

1: Compute an approximate minimum disc cover of P using discs with radius b.
2: Attach each point to the nearest disc center.
3: Treat the centers of discs as terminal points, find a edge length bounded Steiner tree.

To compute a minimum disc cover, there exists polynomial approximation schemes for disc cover using shifting technology in [4].

In 2-dimensional case, the algorithm first partition the bounding box of the input points into squares of side length $l \cdot D$. Then it find optimal covering of points in such a squre by exhaustive search. In such a search, the authors assume that any disk that covers at least two of the input points has two of these points on its border. The algorithm repeats the above process by shifting the partition D distance to the right and get the best covering of all the shifting as the output. The authors proved that this algorithm is an approximation scheme.

Theorem 2 *[4] Let $d \geq 1$ be some finite dimension. Then there is a polynomial time approximation scheme H^d such that for every given natural number $l \geq 1$, the algorithm H^d_l delivers a cover of n given points in a d-dimension Euclidean space by d-dimensional balls of given diameter D in $O(l^d(l \cdot \sqrt{d})^d \cdot (2n)^{d(l\sqrt{d})^{d+1}})$ steps with performance ratio $\leq (1+\frac{1}{l})^d$.*

Using the algorithm in [4], we are able to compute the centers with size no more than $1+\epsilon$ of that of the optimal disc cover. Let $l = \frac{3}{\epsilon}$, then we have

$$\begin{aligned} \text{the performance ratio} &\leq (1+\tfrac{1}{l})^d \\ &= (1+\tfrac{\epsilon}{3})^2 \\ &\leq 1+\epsilon \end{aligned}$$

(We are considering the points in a plane, so $d = 2$ in our problem)

The step 3 of algorithm 1 can be done using the following algorithm 2 from [6]. The input of the algorithm 2 are a set P of n terminals and a given edge bound b. The algorithm outputs a feasible tree T_A spanning P.

Algorithm 2 Minimum Spanning Tree Heuristic

1: Compute a minimum spanning tree T for P.
2: Divide each edge in T into small pieces of length at most b using the minimum number of Steiner points.
3: Output the final tree as T_A.

It is shown in [7, 1] that the performance ratio of algorithm 2 is 4 and tight.

Theorem 3 *[7] The MST heuristic has an approximation factor of $D-1$ in every metric space whose MST number is $D < \infty$.*

MST number is defined as maximum possible degree of a minumum-degree MST spanning points from the space. Since the MST number is 5 for the Euclidean planes, the approximation ratio of MST heuristic is 4.

2.2 Analysis of our algorithm

We analyze the performance ratio of our algorithm with these known results as follows.

Consider an optimal solution T_{OPT} with Steiner point set OPT to our problem. There exists a minimal subset $OPT_1 \subseteq OPT$ that covers all the terminal points, i.e. the discs of radius b centering at points in OPT_1 covers all the terminal points and the size of OPT_1 is minimal among all possible such sets. Let $OPT_2 = OPT \setminus OPT_1$.

Lemma 4 *Let c be a center returned by step 1 of the algorithm 1. Then there must exist a Steiner point $p \in OPT_1$ such that $|cp| \leq 2b$ and vice versa.*

Proof. Consider any terminal point t covered by the disc with c as the center. In an optimal solution, t must be cover by some Steiner point $p \in OPT_1$. Thus $|ct| \leq b$ and $|tp| \leq b$. By triangle inequality, $|cp| \leq 2b$.

On the other direction, consider any terminal point t' covered by a Steiner point $p \in OPT_1$, t' must be covered by some disc with center c'. Note $|pt'| \leq b$ and $|t'c'| \leq b$, we have $|pc'| \leq 2b$ by triangle inequality. □

Theorem 5 *The approximation ratio of the algorithm 1 is $5+\epsilon$.*

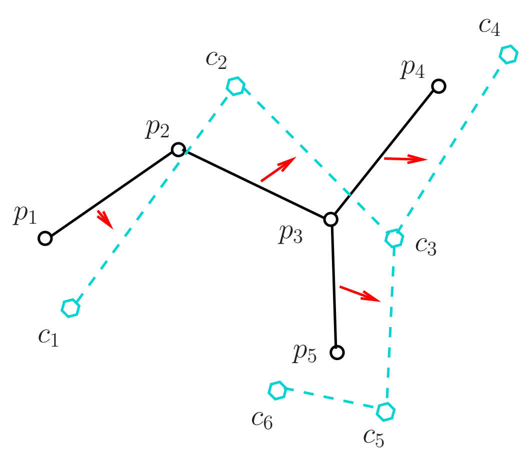

Figure 1: Construct a tree T according to a MST

Proof. Let T' be the steiner tree output by algorithm 2 with OPT_1 as the terminals and b as the bound. Let $S_{T'}$ be the set of steiner points in T'. Let T'_{OPT} be the graph by removing all terminals from T_{OPT}. Note that T'_{OPT} is connected, the vertices set of T'_{OPT} is OPT, and the length of each edge is no more than b. Thus by Theorem 3, $|S_{T'}| \leq 4(|OPT| - |OPT_1|) = 4|OPT_2|$.

Let C be the center of discs returned by step 1. Clearly $|C| \leq (1+\epsilon)|OPT_1|$ since discs with OPT_1 as centers can cover all the terminal points.

Let MST_{OPT_1} be the minimum spanning tree on vertex set OPT_1. We construct a connected graph G_c with vertices set C according to MST_{OPT_1} as follows.

1. For each point $p \in OPT_1$, we pick up a closest point $c \in C$. If there is an edge $p_1p_2 \in MST_{OPT_1}$, we connected c_1 and c_2 (See Figure 1). Let C' be the set of points picked up in this process.

2. For each point $c \in C \setminus C'$, pick up a closest point $c_1 \in C$, connect c and c_1.

Clearly G_c is connected. G_c is not necessary a tree, so we build a tree T_c from G_c by removing long edges from cycles in G_c.

By Lemma 4, we have $|cp| \leq 2b$ in step 1. By triangle inequality, $|c_1c_2| \leq |p_1p_2| + 4b$. Thus the number of degree 2 Steiner points on c_1c_2 will be no more than that on p_1p_2 plus 4.

In step 2, if a point $c \in C$ is not connected, there must be a point $p \in OPT_1$ within distance $2b$ of c by Lemma 4. In step 1, we already picked up a closest point $c' \in C'$ to p. By Lemma 4, $|pc'| \leq 2b$. Thus $|cc'| \leq 4b$ by the triangle inequality.

By triangle inequality, clearly the number of degree 2 Steiner points need to divide tree T_c is no more than

$$\begin{aligned} &\leq |OPT_{T'}| + 4|E_{T_c}| \\ &\leq |OPT_{T'}| + 4(1+\epsilon)(|OPT_1| - 1) \\ &< 4|OPT_2| + 4(1+\epsilon)|OPT_1|. \end{aligned}$$

Since our algorithm build an MST on C, it will output a solution with Steiner points no more than that of T_c

$$\begin{aligned} &\leq (1+\epsilon)|OPT_1| + 4|OPT_2| + 4(1+\epsilon)|OPT_1| \\ &= (5+\epsilon)|OPT_1| + 4|OPT_2| \\ &\leq (5+\epsilon)|OPT|. \end{aligned}$$

□

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