

# Vector Weighted Voronoi Diagrams and Delaunay Triangulations

David Letscher \*

## Abstract

We introduce a weighting scheme for Voronoi diagrams that has preferred directions. This generalizes the concept of weighted Delaunay triangulations and overcomes some of the difficulties of using multiplicative anisotropic weight systems. We discuss properties that make these weighting schemes attractive.

## 1 Introduction

A variety of ideas have been used to generalize Voronoi diagrams and Delaunay triangulations. This include using non-Euclidean metrics; for example  $L_p$  norms [5]. While arbitrary Riemannian metrics may be considered [4], they cannot be dealt with exactly. It is far more practical to have, possibly different, metrics defined for each of the sample points. This can be done by introducing a weight to the “distance” to a point. These weighting systems can either be isotropic, where direction does not influence the metric, or anisotropic where points in some directions appear closer than others. Depending on how the weighting is done, the properties of the corresponding Voronoi diagram and Delaunay triangulation may vary.

In this paper, we introduce an anisotropic weighting scheme that uses vectors as weights and, in some sense, generalizes additive isotropic weights. The triangulations obtained will satisfy some, but not all, of the properties that regular triangulations satisfy [1]. In section 2 existing weighting schemes are discussed with the new anisotropic weighting scheme introduced in section 3. In section 4, properties of the cell decomposition and triangulations obtained will be given. The weighting systems introduced have several generalizations shown in section 5.

## 2 Weighting Systems

There are a several ways that weighting systems have been used to generalize Voronoi diagrams. Each has advantages and disadvantages. Multiplicative schemes tend to be better motivated and additive schemes have better properties.

\*Department of Mathematics and Computer Science, St. Louis University, letscher@slu.edu

If we have a set of points  $X = \{x_1, \dots, x_n\}$  and “metrics”  $\{d_1, \dots, d_n\}$ , then we can use each metric to define how close a point in  $\mathbb{R}^d$  is to the corresponding point in  $X$ . Specifically, the Voronoi diagram will consist of a region  $R_i$  for each point  $x_i$  defined to be

$$R_i = \{p \in \mathbb{R}^d | d_i(p, x_i) \leq d_j(p, x_j) \forall j\}$$

In the unweighted Voronoi diagram, these regions are convex polyhedra and the dual of the Voronoi diagram is the Delaunay triangulation. When the Voronoi diagram is a cell decomposition, the weighted Delaunay triangulation will be the dual of the Voronoi diagram.

Figure 1 has Voronoi diagrams for unweighted points, three existing weighting schemes and the proposed anisotropic weighting. In addition to shown the Voronoi regions, the figures show “circles” of integral radii to illustrate the “metric” for each sample point.

## Multiplicative Weights

Given a point  $p$  and a weight  $w > 0$  then

$$d_w(p, x) = w d(p, x)$$

This defines a metric on  $\mathbb{R}^d$  and has a natural interpretation:  $w$  uniformly scales all distances to the sample point. The intersection of two regions of a Voronoi diagram consists of the points where the distance to one sample point is a multiple of the distance to the other; for example the points that are twice as far from  $p$  as they are from  $q$ . Notice that the faces of the Voronoi diagram with multiplicative weights are non-linear. There are several difficulties with using this weighting system. In particular, the Voronoi diagram for a set of sample points using this weighting system does not necessarily have regions that are simply connected. So the Voronoi diagram may not be cell decomposition as it is in the case of a Euclidean metric. See figure 1(b) for an example.

## “Additive” Weights

Power diagrams provide a form of an additive weighting scheme that addresses some of the issues of multiplicative schemes using the follow notion of “distance”:

$$d_w(p, x) = \sqrt{d^2(p, x) - w}$$

Note that this is not a metric however it can still be used to construct Voronoi diagrams. The motivation is not as obvious in this situation; however, there are several nice properties. For example, points equidistant from two sample points is always a plane. This gives these Voronoi diagrams many of the same properties as standard Voronoi diagrams. Note that some sample points may have empty cells and some points might not be inside of their Voronoi cells. See figure 1(c) for an example.

### (Multiplicative) Anisotropic Weights

Over very small neighborhoods, a good way to approximate an anisotropic metric is using a quadratic form. Given a positive definite matrix  $Q$ :

$$d_Q(p, x) = \sqrt{(x - p)Q(x - p)^T}$$

This defines a metric on  $\mathbb{R}^d$  and if  $Q$  is the identity matrix then this is the usual Euclidean metric. This has been studied by [3]. This metric is similar to using an isotropic multiplicative weighting scheme; the distance is scaled by a constant that depends on direction. “Circle” in this metric are ellipses in the Euclidean metric.

Since multiplicative isotropic weighting is a special case of the anisotropic one, it shares many of the same difficulties. Regions in the Voronoi diagram are some times disconnected and/or not simply connected. This comes from the fact that in this metric the set of points equidistant between two sample points is a conic section, see figure 1(d) for an example. Note that the intersection of two Voronoi cells can be a hyperbola, which forces a region to be disconnected. This problem occur when the metric differs too much between nearby sample points. Points that satisfy extra visibility conditions do not have this difficulty [3].

### 3 Additive Anisotropic Weights

For our weighting scheme every sample point will have an associated vector. This vector essentially forces points along the direction of the vector, and its negative, to be closer than in perpendicular directions. Formally, given a point  $p$  and a vector  $v_p$  we define the weighted distance from a point  $x \in \mathbf{R}^n$  to  $p$  as

$$d_{v_p}(p, x) = \sqrt{\|x - p\|^2 + |v_p \cdot (x - p)|}$$

Note that in directions perpendicular to  $v_p$  this is the Euclidean metric. In directions parallel to  $v_p$ , points that are close in Euclidean space are farther away using this metric. This results in a metric where distance are “squashed” in the direction of  $v_p$ .

To illustrate the metric, figure 2 show “circles” of various radii centered at the point  $p$  with specified weighting vector  $v_p$ . Notice that each circle in this metric is a

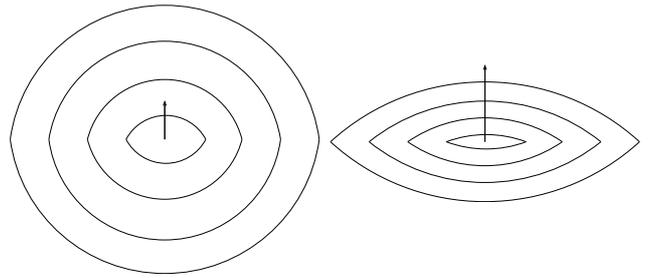


Figure 2: Circles of radius 1,2,3 and 4 centered at  $p$  using the metric  $d_{v_p}$  with  $\|v_p\| = 1$  and  $\|v_p\| = 2$ , respectively.

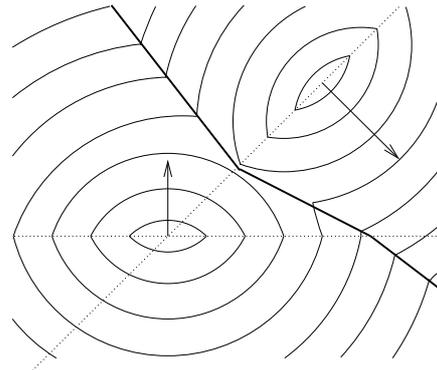


Figure 3: Circles (using weighted anisotropic metrics centered two points and the set of points equidistant from them.

combination of two Euclidean circles that intersect on the axis perpendicular to the weighting vector. As the radius grows larger relative to the length of the vector  $v_p$  they get closer to Euclidean circles. It is the fact that this distortion of the circles reduces as the radius increases that will guarantee several nice properties of this weighting system.

Figure 3 shows how two Voronoi regions intersect. The boundary is piecewise linear with “turns” when it crosses the perpendicular to the weighting vector. It turns out that it crosses one axis at most once which ensures that each Voronoi region is simply connected. Figure 1(d) shows the Voronoi diagram. Note that it has many similarities to the multiplicative anisotropic case with the same sample points and similar weightings. However, each region is connected and simply connect.

### 4 Properties

Voronoi diagrams for Euclidean metrics have many desirable properties. These include having convex regions; which implies that the Voronoi cells are connected and simply connected. Voronoi cells all contain their sample points. And since the set of points equidistant from two

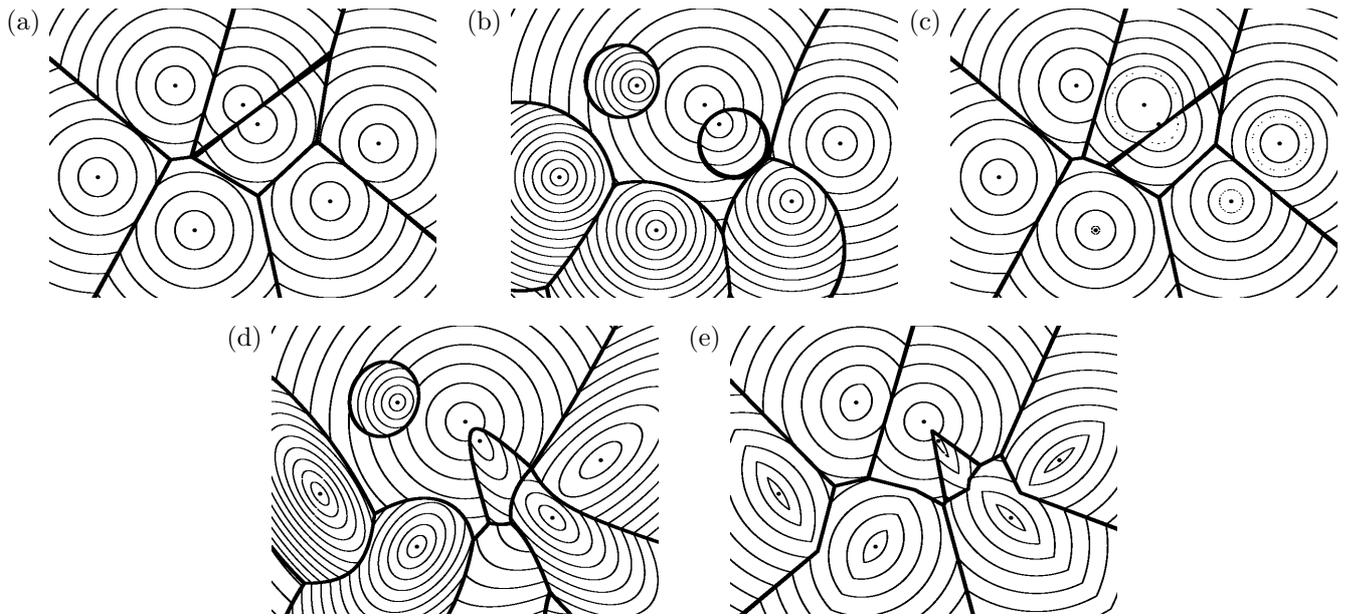


Figure 1: Voronoi diagrams for a set of sample point with (a) no weighting (b) multiplicative weights, (c) “additive” weights, (d) multiplicative anisotropic weights, and (e) vector weights.

sample points is a straight line, the dual of the Voronoi diagram is a triangulation and there is a unique point that is equidistant from three sample points.

Not all of these properties hold for vector weightings. However,  $d_v$  is a metric.

**Lemma 1** *If  $v$  is a vector in the plane then  $d_v$  is a metric. In other words,*

1.  $d_v(x, y) \geq 0$  (non-negativity)
2.  $d_v(x, y) = 0$  if and only if  $x = y$  (identity)
3.  $d_v(x, y) = d_v(y, x)$  (symmetry)
4.  $d_v(x, z) \leq d_v(x, y) + d_v(y, z)$  (triangle inequality)

When either the Euclidean metric or this “vector” metric is used to find the set of points equidistant from a pair of sample points, the quadratic terms cancel out. So solutions to the equation  $d_v(p, x) = d_v(q, x)$  are piecewise linear; it involves linear terms in the coordinates and absolute values of linear terms.

As you move further away from a sample point, the quadratic terms in the metric dominate the term involving the vector. This causes larger “circles” in this metric to be more “round”.

**Lemma 2** *Using the metric  $d_v$ , the ball of radius  $r$  has an aspect ratio of  $\frac{\sqrt{\|v\|^2 + 4r^2} - \|v\|}{2r}$ .*

Note that as  $r$  goes to infinity this aspect ratio approaches 1. This property is one of the reasons that the metric works well.

For standard Voronoi diagrams and ones that use additive weights the set of points equidistant from two sample points is a plane that is perpendicular to the line going through the two points. For multiplicative weights this is a conic section intersecting this line.

When using vector weightings, the dual to the Voronoi diagram will always be a non-Euclidean triangulation, a union of, possibly non-Euclidean, triangles. We will refer to this triangulation as the *vector weighted Delaunay triangulation*.

**Proposition 3** *The Voronoi diagram for a set of points in the plane with vector weights is a cell decomposition and it is dual to a non-Euclidean triangulation.*

One common situation where the dual fails to be a triangulation can be seen in figure 1. There are a pair of Voronoi vertices that are equidistant from the same three sample points. The dual “triangulation” to the Voronoi diagram in figure 1(c) is shown in figure 4. Notice that it has two triangles that are glued together along two edges. This cannot occur for Euclidean triangulations.

An overview of the properties for the various weighting schemes is shown in figure 5. The vector weightings compare well to all of the other methods. The least desirable property that it has is that it can be dual to triangulations that are not realizable as Euclidean ones.

To construct these vector weighted Voronoi diagrams, Fortune’s sweepline algorithm [2] can be modified to find these weighted anisotropic Delaunay triangulation by adding a new event type corresponding to the sweepline hitting “bend” and crosses the axis perpendicular to one

	Euclidean Metric	Multiplicative Weight	Additive Weight	Multiplicative Anisotropic	Vector Weighted
“Metric”	$\ x - y\ $	$c\ x - y\ $	$\sqrt{\ x - y\ ^2 - c}$	$\sqrt{(x - y)Q(x - y)^T}$	$\sqrt{\ x - y\ ^2 +  v \cdot (x - y) }$
Connected Cells	Yes	No	Yes	No	Yes
Convex Cells	Yes	No	No	No	No, Star Convex
Simply Connected Cells	Yes	No	Yes	No	Yes
Sample Point	In cell	In cell	Anywhere	In cell	In cell
Vertices	Unique	Multiple	Unique	Multiple	Unique
Dual	Triangulation	Arbitrary	Arbitrary	Arbitrary	Non-Euclidean triangulation

Figure 5: Property comparison for different metrics for Voronoi diagrams.

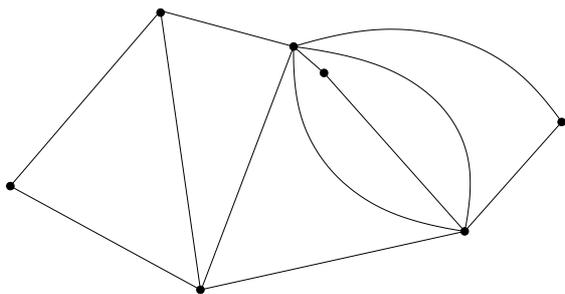


Figure 4: The non-Euclidean triangulation dual to the vector weighted Voronoi diagram in figure 1(e).

of the weight vectors.. Since the separator between two sample points has at most two bends, the number of this new event type is a constant factor of the existing events. This can be used to show that the triangulation can be found in  $O(n \log n)$  time, the same as unweighted Voronoi diagrams.

## 5 Generalizations

There are several ways these weighting schemes can be further generalized. For example multiple vectors can be using in the weighting:

$$d_{\{v_1, \dots, v_l\}}(x, p) = \sqrt{\|x - p\|^2 + \sum_{i=1}^l |v_i \cdot (x - p)|}$$

This will still yield a metric, however, regions of the Voronoi diagram may no longer be connected or simply connected.

Another option would be to incorporate both isotropic and anisotropic weighting simultaneously. A point would then have a pair of a vector and a constant as its weight and the “metric” would be of the form:

$$d_{(v,c)}(x, y) = \sqrt{\|x - y\|^2 + |v \cdot (x - y)| + c}$$

This is not guaranteed to be a cell decomposition and some of the Voronoi regions corresponding to a sample point might be empty.

## 6 Conclusion

We have proposed an anisotropic weighting scheme for Voronoi diagrams and Delaunay triangulations that are easily to calculate and have nice properties for arbitrary point sets. In particular, the Voronoi diagram is always a cell decomposition that is dual to (possibly non-Euclidean) triangulation. The asymptotic running time of the algorithms to construct the diagrams are the same as for ordinary Voronoi diagrams and Delaunay triangulations.

This weighting scheme could be used in anisotropic mesh generation similar to the idea in [3]. The could also have applications in situations where a single vector of information is provided, e.g. the gradient of a function.

## References

- [1] H. Edelsbrunner. *Algorithms in Combinatorial Geometry*. Springer-Verlag, 1987.
- [2] S. Fortune. A sweepline algorithm for Voronoi diagrams. In *SCG '86: Proceedings of the second annual symposium on Computational geometry*, pages 313–322, New York, NY, USA, 1986. ACM Press.
- [3] F. Labelle and J. R. Shewchuk. Anisotropic Voronoi diagrams and guaranteed-quality anisotropic mesh generation. In *Proceedings of the Nineteenth Annual Symposium on Computational Geometry*, pages 191–200, 2003.
- [4] G. Leibon and D. Letscher. Delaunay triangulations and Voronoi diagrams for riemannian manifolds. In *SCG '00: Proceedings of the sixteenth annual symposium on Computational geometry*, pages 341–349, New York, NY, USA, 2000. ACM Press.
- [5] L. Ngoc-Minh. On Voronoi diagrams in the  $l_p$ -metric in higher dimensions. In *STACS '94: Proceedings of the 11th Annual Symposium on Theoretical Aspects of Computer Science*, pages 711–722, London, UK, 1994. Springer-Verlag.