A disk-covering problem with application in optical interferometry

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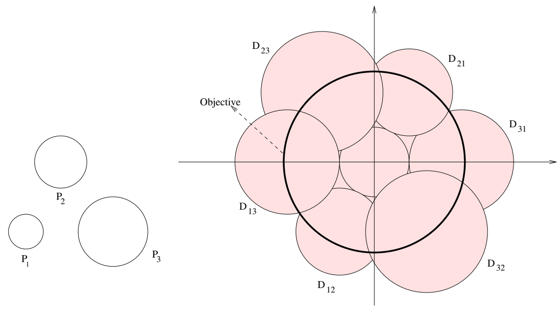

Figure 1: A configuration of three pupils (left) and its ACS (right), the objective is represented by a thick circle.

Abstract

Given a disk O in the plane called the objective, we want to find n small disks $P_1, \ldots, P_n$ called the pupils such that $\bigcup_{i,j=1}^{n} P_i \ominus P_j \supseteq O$, where $\ominus$ denotes the Minkowski difference operator, while minimizing the number of pupils, the sum of the radii or the total area of the pupils. This problem is motivated by the construction of very large telescopes from several smaller ones by so-called Optical Aperture Synthesis. In this paper, we provide exact, approximate and heuristic solutions to several variations of the problem.

1 Introduction

The diameter of the pupil of a telescope is proportional to its resolution power. A simple calculus shows that we would need a telescope having a diameter of approximately $20m$ to observe the Earth from a high orbit [2]. Needless to say, such an instrument would not be adapted to the observation from space. In order not to build too large pupils, Optical Aperture Synthesis is adopted to synthesize (very) large pupils by interferometrically combining several smaller pupils [1]. The auto-correlation support (ACS) of a system of pupils denotes its observable spatial frequency domain (or Fourier domain). When the ACS covers O, we can synthesize a larger pupil by performing a frequency cut-off of the auto-correlation function within O.

Hence the problem can be stated in geometric terms as follows. Given an objective disk O, design a set of disks $\mathcal{P} = \{P_1, \ldots, P_n\}$ such that its ACS $\mathcal{D}$ covers the entire objective while minimizing some cost function. Here $\mathcal{D} = \bigcup_{i,j=1}^{n}(P_i \ominus P_j)$ where $\ominus$ denotes the Minkowski difference operator. The cost function may include the number of pupils, the sum of the radii or the total area of the pupils, etc. Denote by c_i and ρ_i the center and the radius of pupil P_i. Then $\mathcal{D}$ can be written as $\bigcup_{i,j=1}^{n} D_{ij}$ where D_{ij} is the disk of center $c_{ij} = c_i - c_j$ and radius $\rho_i + \rho_j$ (see Fig. 1).

This problem is a variant of the disk-covering problem. To the best of our knowledge, the variant we consider is new and the interferometry problem has not been considered before from a geometric perspective. This paper is a follow-up of our initial investigation [2].

Our results. We give an exact solution for the case of three pupils. For the case where the pupils have the same size, we propose lower and upper bounds of the number of pupils in order to cover the objective. When the centers of the pupils are fixed, using Apollonius diagrams instead of power diagrams as in [2] we obtain an algorithm that minimizes the sum of the radii and provably terminates. Finally, we consider the problem where the radii of the pupils are known but their position is unknown and provide a heuristic algorithm that works well in practice.

2 Problem with three pupils

A configuration of pupils is called valid if its ACS covers the objective. In this section, we want to minimize the sum $\rho_1 + \rho_2 + \rho_3$ among the valid configurations. Let l denote the line passing through c_{23} and c_{32}. Since the disks in $\mathcal{D}$ and the objective are symmetric about the origin, it suffices to consider only one half-plane bounded by l.

Lemma 1 *Among the valid configurations, those in which one radius is half of the objective's radius R and the other two are zero are optimal.*

Proof. We prove by contradiction. It is straightforward to see that such configurations are valid. Consider now a configuration in which $\rho_1 + \rho_2 + \rho_3 < R/2$. We will prove that it cannot be a valid configuration. Indeed, suppose w.l.o.g. P_1 has the largest radius among the three pupils. Then D_{11} is the largest disk among

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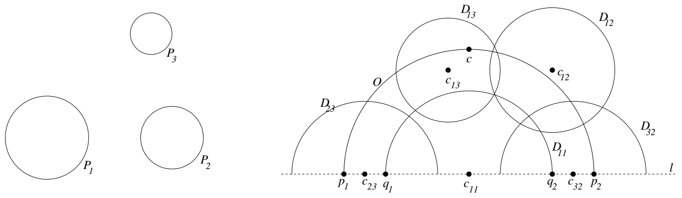

Figure 2: A configuration of three pupils and the upper part of its ACS

D_{11}, D_{22} and D_{33} and its radius $2\rho_1$ is smaller than R. Let p_1, q_1, q_2, p_2 be the intersection points from left to right of ∂O and ∂D_{11} with l (see Fig. 2). If segment $\overline{p_1 q_1}$ is covered by D_{23}, then the diameter of D_{23} is at least the length of $\overline{p_1 q_1}$, i.e., $2(\rho_2 + \rho_3) \geq R - 2\rho_1$ which implies $\rho_1 + \rho_2 + \rho_3 \geq R/2$ (a contradiction). The case where $\overline{p_2 q_2}$ is covered by D_{23} is symmetrical. We can therefore assume that D_{23} does not cover $\overline{p_1 q_1}$ nor $\overline{p_2 q_2}$, and, by symmetry, the same holds for D_{32}. Without loss of generality, we can assume that D_{13} contains p_1 or q_1 and that D_{12} contains p_2 or q_2. We denote by c the midpoint of the arc $p_1 p_2$ of ∂O. The distance of c to p_1, q_1, p_2, q_2 is at least $\sqrt{R^2 + (2\rho_1)^2} > R$. Then c is not included in neither D_{12} nor D_{13} whose diameters are smaller than R. Neither is it included in D_{23} and D_{32} since the distance from c to c_{23} and c_{32} is at least R and the radii of D_{23} and D_{32} are less than R. Hence, the configuration is not valid. □

Lemma 1 points out that using three pupils is not better than just using one.

3 An $8\sqrt{2}$−approximation to the smallest number of pupils of the same radius

In this section, we restrict to the case $\rho_1 = \ldots = \rho_n = \rho/2$, then the disks D_{ij} have the same radius ρ. We want to find an upper bound for n to cover an objective of radius R. As the number of disks is n^2, a lower bound $\lceil R/\rho \rceil$ is easily obtained.

Let p be any prime number, we start by stating a basic result of linear congruences.

Proposition 2 *Let $k, l \in \mathbb{Z}$ such that $\gcd(p, k) = 1$, there exists an integer $0 \leq i < p$ satisfying $ik \equiv l \pmod p$.*

Theorem 3 *$\{x_i - x_j \mid i, j = 0, \ldots, 4p-1\} \supseteq \{x \in \mathbb{Z}, |x| < p^2\}$ where*

$$x_k = kp + (\frac{k(k+1)}{2} \bmod p)$$
$$x_{k+2p} = x_k + p,$$

for $k = 0 \ldots, 2p-1$.

Proof. Let x be an arbitrary integer between 0 and $p^2 - 1$, then x can be written as $kp + l$ for some $0 \leq k, l < p$. We now show that $x = X_i := x_{k+i} - x_i$ for some $i = 0, \ldots, p-1$. Observe that

$$(k-1)p < X_i < (k+1)p, \quad (1)$$

$$X_i \equiv X_0 + ik \pmod p.$$

By Proposition 2 there exists some $0 \leq i < p$ such that $X_i \equiv l \pmod p$. Hence together with (1) the difference of either x_{k+i} or x_{k+i+2p} with x_i will be x. The only case where Proposition 2 does not apply is when $k = 0$. In this case choose $k = 1$ instead and easily see that the set $\{x_{i+1} - x_{i+2p}\} \cup \{x_{i+1+2p} - x_{i+2p}\}$ generates all integers $1, \ldots, p-1$ and hence contains x. □

Suppose, w.l.o.g., that $\rho = \frac{1}{\sqrt{2}}$ and $R = p^2$ for some prime p. Let $\mathcal{S} = \{x \in \mathbb{Z}^2 \mid \|x\|_\infty < p^2\}$. We see that the disks of radius $\frac{1}{\sqrt{2}}$ whose centers cover $\mathcal{S}$ are sufficient to completely cover the objective. In other words, we want to find n centers of pupils $c_i \in \mathbb{Z}^2$ such that

$$\{c_i - c_j \mid 1 \leq i, j \leq n\} \supseteq \mathcal{S}$$

Corollary 4 *$\lceil 8\sqrt{2}R/\rho \rceil$ pupils of radius ρ are sufficient to cover an objective of radius R.*

Proof. The set of pupils is constructed as follows: $c_i = (x_{\lfloor \frac{i}{4p} \rfloor}, x_{i \bmod 4p})$ for $i = 0, \ldots, 16p^2 - 1$. By applying Theorem 3 first for x-coordinate and then for y-coordinate, we see that these $16p^2$ pupils are able to cover any element of $\mathcal{S}$ thus the objective of radius R. As $R = p^2$ and $\rho = \frac{1}{\sqrt{2}}$, we get the upper bound. □

The following is an immediate consequence of Corollary 4 and the lower bound observed earlier.

Corollary 5 *There is an $8\sqrt{2}$−approximation algorithm to cover the objective with the smallest number of pupils with all the same given radius.*

4 Decision problem

Let us recall that D_{ij} has center $c_{ij} = c_i - c_j$ and radius $\rho_i + \rho_j$. We work with the Apollonius diagram of the union of the disks D_{ij}. The (signed) distance of a point x to circle ∂D_{ij} is defined as

$$\delta_{ij}(x) = \|x - (c_i - c_j)\| - (\rho_i + \rho_j).$$

The Apollonius cell of D_{ij} consists of the points whose distance to ∂D_{ij} is no greater than their distance to the other circles in $\mathcal{D}$:

$$A_{ij} = \{x \in \mathbb{R}^2 \mid \delta_{ij}(x) \leq \delta_{i'j'}(x), i', j' = 1, \ldots, n\}.$$

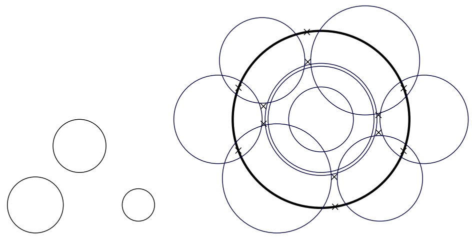

Figure 3: A set of three pupils whose ACS does not cover the objective. The x-marks correspond the vertices of V_{ij} of which some lie outside the union of the disks.

Lemma 6 *The Apollonius cell A_{ij} is included in the disk centered at c_{ij} that contains the set of its vertices.*

Proof. See [3]. □

Let V_{ij} denote the subset of the vertices of A_{ij} that lie inside O plus the intersection points of ∂A_{ij} with ∂O (see Fig. 3). The following lemma states that to cover O, it suffices that each D_{ij} contain V_{ij}, for all i and j. This will be the key idea in our subsequent algorithms.

Lemma 7 *$O \subseteq \mathcal{D}$ iff $V_{ij} \subseteq D_{ij}$ for all $i, j = 1, \ldots, n$.*

Proof. First we argue that $O \subseteq \mathcal{D}$ iff $A_{ij} \cap O \subseteq D_{ij}$ for all $i, j = 1, \ldots, n$. Since the set of A_{ij} forms a decomposition of the plane, $A_{ij} \cap O \subseteq D_{ij}, i, j = 1, \ldots, n$, implies that $O \subseteq \bigcup_{i,j=1}^{n} D_{ij} = \mathcal{D}$. Conversely, suppose that $O \subseteq \mathcal{D}$ and $p \in A_{ij} \cap O$, we will show that $p \in D_{ij}$. Indeed, $p \in O \subseteq \mathcal{D}$ implies $\delta_{\mathcal{D}}(p) \leq 0$. Together with $p \in A_{ij}$, we conclude $\delta_{ij}(p) = \delta_{\mathcal{D}}(p) \leq 0$ which implies that p is inside D_{ij}.

We show next that $A_{ij} \cap O \subseteq D_{ij}$ is equivalent to $V_{ij} \subseteq D_{ij}$ by proving that a disk Δ centered at c_{ij} covering V_{ij} covers also $A_{ij} \cap O$. We first observe that the edges of A_{ij} with both endpoints in O are covered by Δ by Lemma 6. It remains to verify that the intersection points of ∂A_{ij} with ∂O and the arcs linking them are also in Δ. Consider two such points p and q consecutive along the boundary of O. Call $p_1 p_2$ and $q_1 q_2$ the two Apollonius edges that intersect ∂O at p and q respectively. Suppose $p_1, q_1 \in O$ and $p_2, q_2 \notin O$, which implies that p_1, p, q, q_1 belong to V_{ij}. Since p_1 and p lie on edge $p_1 p_2$, and q and q_1 are contained in $q_1 q_2$, Δ will cover the arcs $p_1 p$ and $q q_1$ by Lemma 6. It thus remains to show that the circular arc pq of O is included in Δ, which is true since $p, q \in D_{ij}$ whose radius has been assumed to be smaller than the radius of O. □

5 Minimizing the sum of the radii of the pupils

In this section the pupil centers are fixed and we consider the problem of optimizing the sum of the radii of the pupils.

We start with an initial configuration and compute the sets V_{ij}. Let ρ_{ij} be the smallest radius for D_{ij} to cover V_{ij}. If we change the radii of the pupils so that the radius of each D_{ij} is at least ρ_{ij} then the objective is covered according to Lemma 7. We hence need to solve the following linear program. See [2] for additional linear constraints.

$$\begin{array}{lll} \min & \sum_{i=1}^{n} \rho'_i & \\ \text{s.t.} & \rho'_i + \rho'_j \geq \rho_{ij}, & i, j = 1, \ldots, n \quad (*) \\ & \rho'_i \geq 0, & i = 1, \ldots, n. \end{array}$$

We may iterate the above steps until the sum of the radii converges. Observe that the Apollonius diagram of the disks needs to be updated at each iteration since the pupils' radii change.

Algorithm 1 Minimize the sum of the pupils' radii

```
ε ← any small positive constant
repeat
    construct the Apollonius diagram of 𝒟
    compute V_ij and ρ_ij
    compute {ρ'_i}_{1≤i≤n} by solving linear program (*)
    err ← Σ_{i=1}^n ρ_i − Σ_{i=1}^n ρ'_i
    ρ_i ← ρ'_i, i = 1, ..., n
until err < ε except for the first iteration
return {ρ'_i}_{1≤i≤n}
```

Lemma 8 *Algorithm 1 always terminates.*

Proof. The initial V_{ij} is included in D'_{ij} by the construction of α_{ij}. According to Lemma 7, O is covered by $\bigcup_{ij} D_{ij}$ after the first iteration. Hence, we may assume that the objective is covered. In this case, no α_{ij} is positive which shows that, at each step, $\rho'_i + \rho'_j \leq \rho_i + \rho_j$ and hence $\sum_{i=1}^{n} \rho'_i \leq \sum_{i=1}^{n} \rho_i$. Since $\sum_{i=1}^{n} \rho'_i$ is positive, Algorithm 1 necessarily terminates after a finite number of iterations. □

Algorithm 1 has been implemented and appears to work well in practice. Fig. 4 compares the results of Algorithms 1 with the optimal solution computed by the exhaustive search method (see [3]).

6 Fixed-radius problem

In this section, we fix the radii and propose a heuristic algorithm for moving the set of pupils in order that its ACS covers O. Our algorithm works as follows. We begin with a given configuration of pupils and compute the sets V_{ij}. Desiring that each D_{ij} should contain V_{ij}

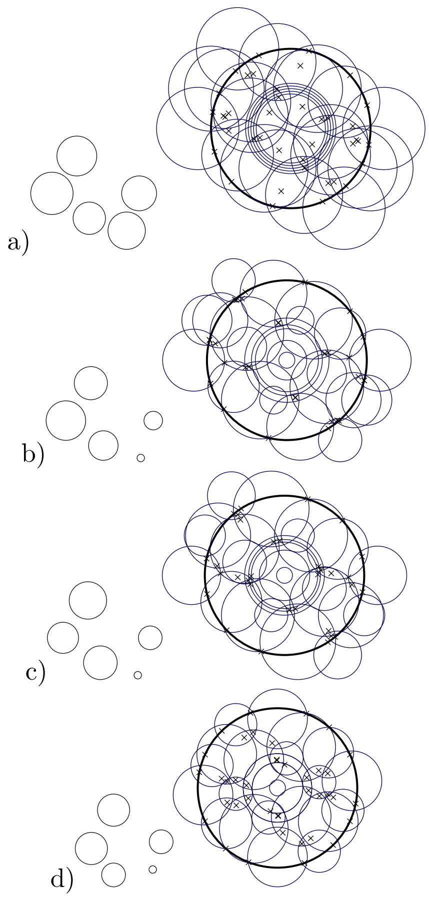

Figure 4: Initial configuration of 5 pupils (a). Results after applying Algorithm 1 (b) and the exhaustive search algorithm (c). The total areas of the pupils in (a) and (b) are 35.657 and 19.842 respectively. The optimal solution must be at least 19.572 as computed by the exhaustive algorithm. The area of the pupils in (d) is only 16.480 when we move the pupils by the algorithm described in Section 6 and then apply Algorithm 1.

for all i and j, we minimize the sum of the squared distance of the centers of D_{ij} to the points in V_{ij}.

$$\min \sum_{i,j=1}^{n} \sum_{v \in V_{ij}} \|(c'_i - c'_j) - v\|^2$$

Here centers c'_i of the pupils are variables and we recall that $c'_i - c'_j$ would become the center of disk D'_{ij}. The objective function being the sum of convex functions, is thus convex. We can update the sets V_{ij} and iterate the algorithm until we obtain the desired result. Our algorithm most of the time provides a valid solution if one exists, irrespectively of the initial configuration of pupils (see Figs. 5 and 6). The algorithm can also be used as a preprocessing step to improve Algorithm 1 (see Fig. 4d).

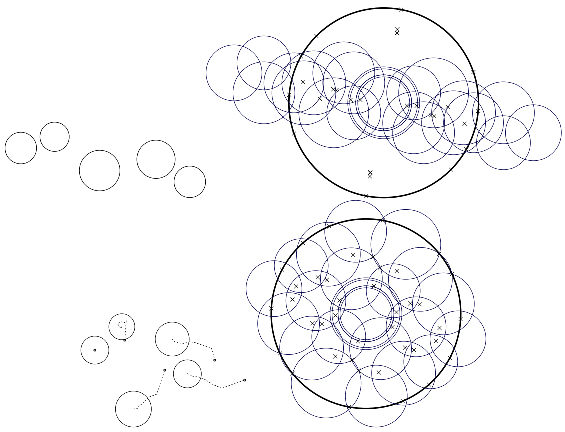

Figure 5: Left: A configuration of 5 pupils whose centers are initially placed about a line. b) Right: Dotted curves illustrate the movements of the pupils after iterating 24 times the algorithm until the union of the disks covers O.

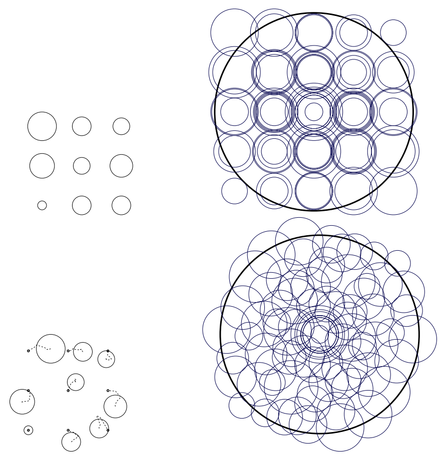

Figure 6: Left: A configuration of 9 pupils Right: Result obtained by iterating 9 times the algorithm.

References

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