Approximate Shortest Descent Path on a Terrain

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Abstract

A path from a point s to a point t on the surface of a polyhedral terrain is said to be *descent* if for every pair of points $p = (x(p), y(p), z(p))$ and $q = (x(q), y(q), z(q))$ on the path, if $dist(s, p) < dist(s, q)$ then $z(p) \geq z(q)$, where $dist(s, p)$ denotes the distance of p from s along the aforesaid path. Although an efficient algorithm to decide if there is a descending path between two points is known for more than a decade, no efficient algorithm is yet known to find a shortest descending path from s to t in a polyhedral terrain. In this paper we propose an $(1 + \epsilon)$-approximation algorithm running in polynomial time for the same.

1 Introduction

The problem of finding *descending paths* in a polyhedral terrain was first studied by Berg and Kreveld [4]. They presented an $O(n \log n)$ time algorithm to decide if there is a descending path between two points where n is the number of faces of the triangulated terrain. Roy et al. [6] presented an $O(n^2 \log n)$ time algorithm to compute a shortest descending path (SDP) in a convex terrain, and an $O(n \log n)$ time algorithm to compute an SDP through a sequence of parallel edges. Recently, Ahmed and Lubiw [1] have proposed an $(1 + \epsilon)$-approximation algorithm for SDP over a given face sequence using convex optimization techniques. The running time of their algorithm is $O(n^{3.5} \log(\frac{1}{\epsilon}))$. Motivation for studying this problem is discussed in [1, 4, 6].

In this paper we propose an $(1 + \epsilon)$-approximation algorithm for SDP in a polyhedral terrain. The time complexity of our proposed algorithm is $O(mn \log mn + nm^2)$ time, where n is the number of faces of the triangulated terrain and m is approximately $O(\frac{n}{\epsilon} \log \frac{1}{\epsilon})$ ignoring the geometric parameters (see Claim 2). Thus we are able to provide the first polynomial time approximation algorithm for the open question raised in [4]. We make use of the discretization method of Aleksandrov et al. [3] that was used to solve the shortest path problem in a weighted terrain. The method involves inserting Steiner points on the edges of the terrain and then constructing an edge weighted graph G. Finally, the problem boils down to finding a shortest path between two points in G.

One of the reviewers of this paper has pointed out a very recent technical report [2] by Ahmed and Lubiw that proposes an $(1 + \epsilon)$-approximation algorithm for SDP in a polyhedral terrain. It discretizes the terrain with $O(n^2 \frac{X}{\epsilon})$ Steiner points so that after an $O(n^2 \frac{X}{\epsilon} \log(\frac{nX}{\epsilon}))$-time preprocessing phase for a given vertex s, it can report an $(1+\epsilon)$-approximate path from s to any point v in $O(n)$ time if v is either a vertex of a terrain or a Steiner point, and in $O(\frac{nX}{\epsilon})$ time otherwise. Note that $X = \frac{L}{h} \sec \theta$, where L is the length of the longest edge, h is the smallest distance of a vertex from a non-adjacent edge in the same face, and θ is the largest acute angle between a non-level edge and a perpendicular line.

2 Preliminaries

A terrain $\mathcal{T}$ is a polyhedral surface in $\mathbb{R}^3$ with a special property: the vertical line at any point on the xy-plane intersects the surface of $\mathcal{T}$ at most once. Thus, the projections of all the faces of a terrain on the xy-plane are mutually non-intersecting at their interior. Each vertex p of the terrain is specified by a triple $(x(p), y(p), z(p))$. Without loss of generality, we assume that all the faces of $\mathcal{T}$ are triangles, and the source s and destination t are the vertices of the terrain. Our aim is to find an $(1 + \epsilon)$-approximate SDP in $\mathcal{T}$ from s to t. Our algorithm works in three phases.

Phase 1: In the first phase, we find out the descent flow region for the source s using the method that was described in [6]. Given an arbitrary point p on the surface of the terrain $\mathcal{T}$, the descent flow region of p (called $DFR(p)$) is the region on the surface of $\mathcal{T}$ such that each point $q \in DFR(p)$ is reachable from p through a descent path. Then we check whether t lies inside $DFR(s)$ is true. If the answer is negative, then descent path from s to t does not exist. If the answer is positive, then we proceed to phase 2. Note that, like $\mathcal{T}$, $DFR(s)$ is also triangulated.

Phase 2: In this phase we use a method similar to the one that was employed by Aleksandrov et al. [3] and construct a graph G. We deviate from [3] on two accounts: We insert two sets of Steiner points on the

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edges of the $\mathcal{T}$ instead of one. Of these two sets, the first set of Steiner points is same as that of [3]. The second difference is that, unlike in [3], the graph edges are directed. We discuss these details in the next section.

Phase 3: We use Dijkstra's algorithm [5] using Fibonacci heaps to find the shortest path between s and t in the directed graph G. We later show that this path is, in fact, an $(1+\epsilon)$-approximation to SDP.

3 Graph G

3.1 Steiner Points Insertion

We insert two different sets of Steiner points on the edges of the $DFR(s)$.

ϵ-Steiner-points: This set of Steiner points is the same as that of [3]. Let v be a vertex of $DFR(s)$. Define h_v to be the minimum distance from v to the boundary of the union of its incident faces. Let $r_v = \epsilon h_v$ for some $\epsilon > 0$.

For each vertex v of face f_i, do the following: Let e_q and e_p be the edges of f_i incident to v. First, place Steiner points on the edges e_q and e_p at distance r_v from v; call them q_1 and p_1, respectively. By definition, $|vq_1| = |vp_1| = r_v$. Let θ_v be the angle between e_p and e_q. Define

$$\delta = \begin{cases} (1+\epsilon.\sin\theta_v) & \text{if } \theta_v < \frac{\pi}{2}, \\ \delta = (1+\epsilon) & \text{otherwise.} \end{cases}$$

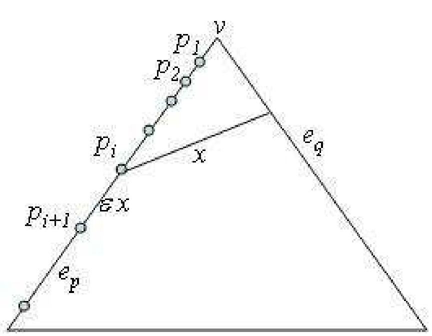

Figure 1: *ϵ-Steiner Points*

We now add Steiner points $q_2, q_3, \ldots, q_{\mu_q-1}$ along e_q such that $|vq_j| = r_v\delta^{j-1}$ where $\mu_q = \lceil \log_\delta(\frac{|e_q|}{r_v})\rceil$. Similarly, add Steiner $p_2, p_3, \ldots, p_{\mu_p-1}$ along e_p where $\mu_p = \lceil \log_\delta(\frac{|e_p|}{r_v})\rceil$. See Figure 1.

Define $dist(a,e)$ as the minimum distance from a point a to an edge e. This segment from a to e will be perpendicular to e.

Claim 1 *(3.11 of [3]) $|q_iq_{i+1}| \leq \epsilon.dist(q,e_p)$ and $p_jp_{j+1} \leq \epsilon.dist(p,e_q)$ where $0 < i < \mu_q$, $0 < j < \mu_p$, $q \in q_iq_{i+1}$ and $p \in p_jp_{j+1}$.*

Isohypse-Steiner-points: Let $p_1, p_2, \ldots, p_{\mu_q}$ be the set of ϵ-Steiner-points on some edge e_p. For any non-horizontal edge (An edge e is horizontal if for any two points a and b on the edge e, $z(a) = z(b)$) $e_j(\neq e_i)$ add an Isohypse-Steiner-point dp_i on edge e_j if $z(dp_i) = z(p_i)$. So, if there is no point on edge e_j such that $z(dp_i) = z(p_i)$, then no Isohypse-Steiner-point is inserted. Intuitively, we take a horizontal plane at each Steiner point and intersect it with the terrain. At each intersection of the plane and an edge of terrain we insert a Steiner point. Note that, the insertion of Isohypse-Steiner-point may increase the total number of ϵ-Steiner-points by at most a factor of n.

3.2 Graph Construction

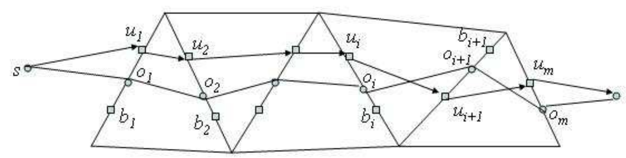

Figure 2: *Approximation of SDP*

For each face f_i we treat all the ϵ-Steiner-points, Isohypse-Steiner-points and the vertices of the $DFR(s)$ as the node of graph G_i. Two vertices a and b of G_i are connected by a directed edge e_{ab} if a and b lie on two different edges of face f_i and $z(a) \geq z(b)$. The weight of each edge is the Euclidean distance between a and b. Now we define an edge weighted directed graph $G = G_1 \cup G_2 \cup \ldots \cup G_n$.

Claim 2 *At most $m = O(n\log_\delta(\frac{|L|}{r}))$ Steiner points are added to each edge of f_i, for $1 \leq i \leq n$, and where $|L|$ is the length of the longest edge in $DFR(s)$ and r is the minimum among the r_v.*

Claim 3 *G has $O(n^2\log_\delta(\frac{|L|}{r}))$ vertices and $O(n^3\log^2_\delta(\frac{|L|}{r}))$ edges.*

4 Proof of Approximation Factor

Let $\pi(s,t) = [s, o_1, o_2, \ldots, o_m, t]$ be the shortest monotone descent path and it passes through the edge sequence $e_1, e_2, \ldots, e_m$. Then in each edge e_i the path $\pi(s,t)$ must pass through a pair of Steiner-points which are closest to o_i, say u_i and b_i (See Figure 2) such that $z(u_i) \geq z(o_i) \geq z(b_i)$. Note that, in degenerate case point u_i may be o_i and o_i may coincide with b_i. Let us consider the path $\pi^*(s,t) = [s, u_1, u_2, \ldots, u_m, t]$.

Lemma 4 *$\pi^*(s,t)$ is monotone descent.*

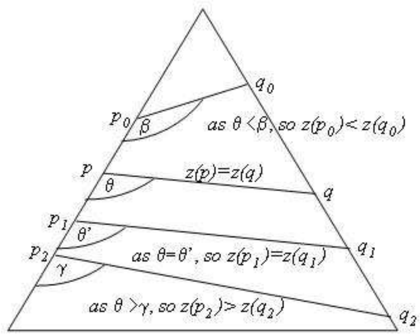

Figure 3: *Illustration of Lemma 5 and Corollary 6*

To prove the above Lemma we need to prove the following statement.

Lemma 5 *Let p and q be two points on two consecutive members e_i and e_{i+1} of $\mathcal{E}$ which bounds a face f, and $z(p) = z(q)$. Now, if a line ℓ on face f intersects both e_i and e_{i+1}, and is parallel to the line segment $[p, q]$, then all the points on ℓ have the same z-coordinate.*

Proof. Consider a horizontal plane h at altitude $z(p)$. The intersection of the face f and the plane h is the line segment $[p, q]$. Consider another horizontal plane h' through a point r on line ℓ. The intersection of f and h' must be line parallel to $[p, q]$, and hence it coincides with the line ℓ. Thus, all the points on the line ℓ have the same z-coordinate. □

From the above Lemma, we get the following Corollary.

Corollary 6 *If $[p, q]$ makes an angle θ with e_i (in anticlockwise direction see Figure 3) and $[v, w]$ ($v \in e_i$ and $w \in e_{i+1}$) be the line segment that makes an angle β with e_i (in anticlockwise direction). Now*

1. *$\theta = \beta$ iff $z(v) = z(w)$,*
2. *$\theta > \beta$ iff $z(v) > z(w)$, and*
3. *$\theta < \beta$ iff $z(v) < z(w)$.*

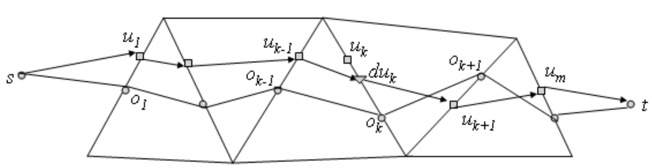

Figure 4: *Proof of Lemma 4*

Now we have to prove that the path $\pi^*(s,t)$ is monotone descent. Without loss of generality we can say that $z(s) \geq z(u_1)$. We will prove this using contradiction. So the path from s to u_1 is monotone descent. Let us consider that path s to u_{k-1} along $\pi'(s,t)$ is monotone descent and the path segment u_{k-1} to u_k is not monotone descent. Then $z(u_{k-1}) < z(u_k)$. Again the path segment o_{k-1} to o_k is monotone descent and $z(u_k) > z(o_k)$ (As $z(u_k) > z(b_k)$). So there is a Isohypse-Steiner-point $du_k \in [u_k, o_k]$ (by Lemma 5 and Corollary 6 see Figure 4) such that path from s to du_k is monotone descent. This contradicts the assumption that u_k is closet to o_k.

From the above discussion, we can conclude that the path $\pi^*(s,t)$ is monotone descent. Hence Lemma 4 follows.

Lemma 7 *$|\pi^*(s,t)| \leq (1 + 2\epsilon)|\pi(s,t)|$, where $|\pi(a,b)|$ implies the length of the path $\pi(a,b)$.*

Proof. We have

$$\begin{aligned}
|\pi^*(s,t)| &= |su_1| + |u_1u_2| + |u_2u_3| + \ldots + \\
&\quad +|u_{m-1}u_m| + |u_mt| \\
&\leq |so_1| + |o_1u_1| + |u_1o_1| + |o_1o_2| + \\
&\quad +|o_2u_2| + |u_2o_2| + |o_2o_3| + \\
&\quad +|o_3u_3| + |u_3o_3| + \ldots + |o_{m-1}u_{m-1}| \\
&\quad +|o_{m-1}o_m| + |o_mu_m| + |u_mo_m| + |o_mt| \\
&= |so_1| + |o_1o_2| + |o_2o_3| + \ldots + \\
&\quad +|o_{m-1}o_m| + |o_mt| + \\
&\quad +2\{|o_1u_1| + |o_2u_2| + \ldots + |u_mo_m|\} \\
&\leq |so_1| + |o_1o_2| + |o_2o_3| + \ldots + \\
&\quad +|o_{m-1}o_m| + |o_mt| + \\
&\quad +2\epsilon\{|o_1o_2| + |o_2o_3| + \ldots + |o_{m-1}o_m|\} \\
&\quad \text{(using Claim 1, Lemma 5 and Corollary 6)} \\
&\leq (1 + 2\epsilon)\pi(s,t).
\end{aligned}$$

□

From the above Lemma we can conclude that there exists a path $\pi^*(s,t)$ such that $|\pi^*(s,t)| = (1 + 2\epsilon)|\pi(s,t)|$. Dijkstra's algorithm may output a monotone descent path $\pi^{**}(s,t)$ which may be a different path from $\pi^*(s,t)$. As Dijkstra's algorithm outputs a path of shortest length in graph G, we have $|\pi^*(s,t)| \geq |\pi^{**}(s,t)|$. So, we can conclude the following theorem.

Theorem 8 *Let $0 < \epsilon < \frac{1}{2}$. Let $DFR(s)$ be a terrain with n faces and let s and t be two of its vertices. An approximation $\pi'(s,t)$ of a shortest monotone path $\pi(s,t)$ can be computed such that $\pi'(s,t) \leq (1+2\epsilon)\pi(s,t)$. The approximation can be computed in $O(mn \log mn + nm^2)$ time where $m = O(n \log_\delta(\frac{|L|}{r}))$ and $\delta = (1 + \epsilon \sin\theta)$ (by virtue of Claim 3), when θ, L, r denote the smallest angle among the triangles, the length of the longest edge, and the length of smallest r_v, respectively.*

Proof. The approximation factor follows from Lemma 7. The running time of the algorithm is same as that of single source shortest path algorithm on a graph with $O(n^2 \log_\delta(\frac{|L|}{r}))$ vertices and $O(n^3 \log_\delta^2(\frac{|L|}{r}))$ edges. $\square$

References

[1] M. Ahmed and A. Lubiw, Shortest descending paths through given faces, *Proc. Canadian Conference on Computational Geometry*, pp. 35-38, 2006.

[2] M. Ahmed and A. Lubiw, An Approximation Algorithm for Shortest Descending Paths, *University of Waterloo Technical Report CS-2007-14*, May 2007, http://arxiv.org/abs/0705.1364v1.

[3] L. Aleksandrov, M. Lanthier, A. Maheshwari and J. R. Sack, An ϵ-Approximation for Weighted Shortest Paths on Polyhedral Surfaces, *Proc. Scandinavian Workshop on Algorithm Theory*, pp. 11-22, 1998.

[4] M. de Berg and M. van Kreveld, Trekking in the Alps without freezing or getting tired, *Algorithmica*, vol. 18, pp. 306-323, 1997.

[5] E.W. Dijkstra, A Note on Two Problems in Connection with Graphs, *Numerical Mathematics*, vol. 1, pp. 269-271, 1959.

[6] S. Roy, S. Das and S. C. Nandy, Shortest monotone descent path problem in polyhedral terrain, *Proc. Symp. on Theoretical Aspects of Computer Science*, pp. 281-292, 2005.