Approximate Shortest Descent Path on a Terrain

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Abstract

A path from a point s to a point t on the surface of a polyhedral terrain is said to be *descent* if for every pair of points $p = (x(p), y(p), z(p))$ and $q = (x(q), y(q), z(q))$ on the path, if $dist(s, p) < dist(s, q)$ then $z(p) \geq z(q)$, where $dist(s, p)$ denotes the distance of p from s along the aforesaid path. Although an efficient algorithm to decide if there is a descending path between two points is known for more than a decade, no efficient algorithm is yet known to find a shortest descending. In this paper we propose an $(1 + \epsilon)$-approximation algorithm running in polynomial time for the same.

1 Introduction

The problem of finding *descending paths* in a polyhedral terrain was first studied by de Berg and van Kreveld [5]. They presented an $O(n \log n)$ time algorithm to decide if there is a descending path between two points where n is the number of faces of the triangulated terrain. Roy et al. [7] presented an $O(n^2 \log n)$ time algorithm to compute a shortest descending path (SDP) in a convex terrain, and an $O(n \log n)$ time algorithm to compute a SDP through a sequence of parallel edges. Recently, Ahmed and Lubiw [1] have proposed an $(1 + \epsilon)$-approximation algorithm for computing a SDP over a given face sequence using convex optimization techniques. The running time of their algorithm is $O(n^{3.5} \log(\frac{1}{\epsilon}))$. Motivation for studying this problem is discussed in [1, 5, 7].

In this paper we propose an $(1+\epsilon)$-approximation algorithm for computing a SDP in a polyhedral terrain. We make use of the discretization method of Aleksandrov et al. [3] that was used to solve the shortest path problem in a weighted terrain. The method involves inserting Steiner points on the edges of the terrain and then constructing an edge weighted graph G. Finally, the problem boils down to finding a shortest path between two points in G. We propose two algorithms for SDP problem. The first one uses Dijkstra's algorithm [6] to compute shortest paths and runs in $O(|V| \log |V| + |E|)$ time; the second one uses Bushwhack algorithm [8] and runs in $O(|V| \log |V|)$ time, where $|V| = O(n^2 \log_\delta(\frac{n|L|}{r}))$ and $|E| = O(n^3 \log^2_\delta(\frac{n|L|}{r}))$. Here $\delta = (1 + \epsilon sin\theta)$ and θ, L, r denote the smallest angle among the triangles, the length of the longest edge, and the length of smallest r_v, respectively (see Claim 3 for more details).

Recently, Ahmed and Lubiw [2] have proposed an $(1+\epsilon)$-approximation algorithm for computing SDP in a polyhedral terrain. It discretizes the terrain with $O(n^2 \frac{X}{\epsilon})$ Steiner points so that after

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an $O(n^2 \frac{X}{\epsilon} \log(\frac{nX}{\epsilon}))$-time preprocessing phase for a given vertex s, they can report an $(1+\epsilon)$-approximate path from s to any point v in $O(n)$ time if v is either a vertex of a terrain or a Steiner point, and in $O(\frac{nX}{\epsilon})$ time, otherwise. The quantity $X = \frac{L}{h} \sec\theta$, where L is the length of the longest edge, h is the smallest distance of a vertex from a non-adjacent edge in the same face, and θ is the largest acute angle between a non-level edge and a perpendicular line [2].

2 Preliminaries

A terrain $\mathcal{T}$ is a polyhedral surface in $I\!R^3$ with a special property: the vertical line at any point on the xy-plane intersects the surface of $\mathcal{T}$ at most once. Thus, the projections of all the faces of a terrain on the xy-plane are mutually non-intersecting at their interior. Each vertex p of the terrain is specified by a triple $(x(p), y(p), z(p))$. Without loss of generality, we assume that all the faces of $\mathcal{T}$ are triangles, and the source s and destination t are the vertices of the terrain. Our aim is to find an $(1+\epsilon)$-approximate SDP in $\mathcal{T}$ from s to t.

Let v be vertex of $\mathcal{T}$. Let h_v be the minimum distance to the opposite boundary of the adjacent faces a v. Define a polygonal cap C_v, called a sphere around v, as follows. Let $r_v = \epsilon h_v$ and $r'_v = \frac{\epsilon h_v}{n}$ for some $\epsilon > 0$. Let r' be the minimum among the r'_v over all v. Let uvw be a triangular face incident to v and v' (w') be at the distance r'_v from v on edge the uv (wv). Then $u'vw'$ is a triangular sub-face inside uvw. The sphere C_v around v contains all such subface incident at v.

Property 1 *[3] Let v_a and v_b two vertices of the terrain $\mathcal{T}$. The geodesic distance between any two spheres C_{v_a} and C_{v_b} is greater than $(1-2\epsilon)h_{v_a}$.*

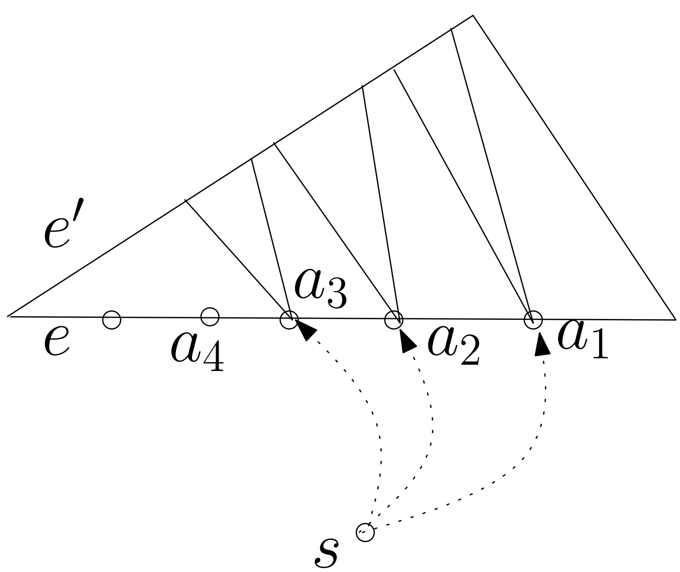

Figure 1: *The Bushwack Algorithm*

The Bushwhack algorithm, proposed by Sun and Reif [8], computes a shortest path in the geometric graph defined over Steiner points in a terrain. It explores geometric properties of the paths in such graphs to attain an $O(|V| \log |V|)$ running time, instead of $O(|V| \log |V| + |E|)$ as in Dijkstra's single source shortest path algorithm. Here V and E refers to vertices and edges of the underlying graph, respectively. The Bushwhack algorithm uses the following simple property of shortest paths on the terrains: Let p_1 and p_2 are two shortest paths. The paths p_1 and p_2 do not intersect each other at an interior point of any face.

Therefore, for any two consecutive Steiner points a_1 and a_2 on an edge e for which the distance from s are already known, the corresponding sets of possible next nodes on the path are disjoint (see Figure 1). This enable us to consider only a subset of links at a Steiner point u when expanding the shortest path tree from u.

3 Roadmap of our algorithm

Our algorithm works in three phases.

Phase 1: In the first phase, we find out the descent flow region for the source s using the method that was described in [7]. Given an arbitrary point p on the surface of the terrain $\mathcal{T}$, the descent flow region of p (called $DFR(p)$) is the region on the surface of $\mathcal{T}$ such that each point $q \in DFR(p)$ is reachable from p through a descent path. Then we check whether t lies inside $DFR(s)$ is true. If the answer is negative, then descent path from s to t does not exist. If the answer is positive, then we proceed to phase 2. Note that, like $\mathcal{T}$, $DFR(s)$ is also triangulated.

Phase 2: In this phase we use a method similar to the one that was employed by Aleksandrov et al. [3] and construct a graph G. We deviate from [3] on two accounts: We insert two sets of Steiner points on the edges of the $\mathcal{T}$ instead of one. Of these two sets, the first set of Steiner points is same as that of [3]. The second difference is that, unlike in [3], the graph edges are directed. We discuss these details in the next section.

Phase 3: We use Dijkstra's algorithm [6] or Bushwhack algorithm [8] to find the shortest path between s and t in the directed graph G. We later show that this path is, in fact, an $(1+\epsilon)$-approximation to SDP.

4 Approximated Graph G

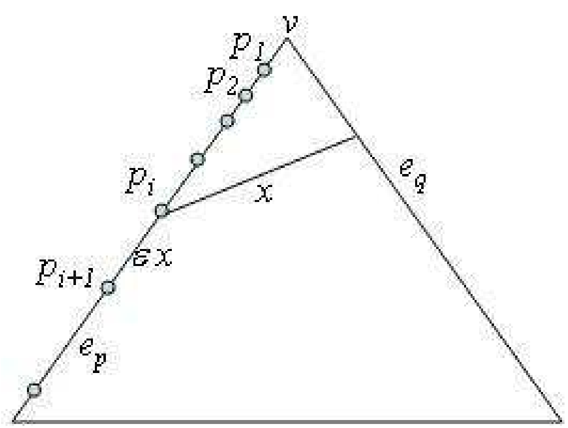

Figure 2: *ϵ-Steiner Points*

4.1 Steiner Points Insertion

We insert two different sets of Steiner points on the edges of the $DFR(s)$.

ϵ-Steiner-points: This set of Steiner points is the same as that of [3]. Let v be a vertex of $DFR(s)$. Define h_v to be the minimum distance from v to the boundary of the union of its incident faces. Let $r_v = \epsilon h_v$ and $r'_v = \frac{\epsilon h_v}{n}$ for some $\epsilon > 0$.

For each vertex v of face f_i, do the following: Let e_q and e_p be the edges of f_i incident to v. First, place Steiner points on the edges e_q and e_p at distance r'_v from v; call them q_1 and p_1, respectively. By definition, $|vq_1| = |vp_1| = r'_v$. Let θ_v be the angle between e_p and e_q. Define

$$\delta = \begin{cases} (1 + \epsilon . \sin\theta_v) & \text{if } \theta_v < \frac{\pi}{2}, \\ (1 + \epsilon) & \text{otherwise.} \end{cases}$$

We now add Steiner points $q_2, q_3, \ldots, q_{\mu_q - 1}$ along e_q such that $|vq_j| = r'_v \delta^{j-1}$ where $\mu_q = \lceil \log_\delta(\frac{|e_q|}{r'_v}) \rceil$. Similarly, add Steiner $p_2, p_3, \ldots, p_{\mu_p - 1}$ along e_p where $\mu_p = \lceil \log_\delta(\frac{|e_p|}{r'_v}) \rceil$. See Figure 2.

Define $dist(a, e)$ as the minimum distance from a point a to an edge e. This segment from a to e will be perpendicular to e.

Claim 1 *(3.11 of [3])* $|q_i q_{i+1}| \leq \epsilon . dist(q, e_p)$ *and* $p_j p_{j+1} \leq \epsilon . dist(p, e_q)$ *where* $0 < i < \mu_q$, $0 < j < \mu_p$, $q \in q_i q_{i+1}$ *and* $p \in p_j p_{j+1}$.

Isohypse-Steiner-points: Let $p_1, p_2, \ldots, p_{\mu_q}$ be the set of ϵ-Steiner-points on some edge e_p. For any non-horizontal edge (an edge e is horizontal if for any two points a and b on the edge e, $z(a) = z(b)$) $e_j (\neq e_i)$ add an Isohypse-Steiner-point dp_i on edge e_j if $z(dp_i) = z(p_i)$. So, if there is no point on edge e_j such that $z(dp_i) = z(p_i)$, then no Isohypse-Steiner-point is inserted. Intuitively, we take a horizontal plane at each Steiner point and intersect it with the terrain. At each intersection of the plane and an edge of terrain we insert a Steiner point. Note that, the insertion of Isohypse-Steiner-point may increase the total number of ϵ-Steiner-points by at most a factor of n.

4.2 Graph Construction

For each face f_i we treat all the ϵ-Steiner-points, Isohypse-Steiner-points and the vertices of the $DFR(s)$ as the node of graph G_i. Two vertices a and b of G_i are connected by a directed edge e_{ab} if a and b lie on two different edges of face f_i and $z(a) \geq z(b)$. The weight of each edge is the Euclidean distance between a and b. Now we define an edge weighted directed graph $G = G_1 \cup G_2 \cup \ldots \cup G_n$.

Claim 2 *At most* $m = O(n \log_\delta(\frac{n|L|}{r}))$ *Steiner points are added to each edge of* f_i, *for* $1 \leq i \leq n$, *and where* $|L|$ *is the length of the longest edge in* $DFR(s)$ *and* r *is the minimum among the* r_v.

Claim 3 G *has* $|V| = O(n^2 \log_\delta(\frac{n|L|}{r}))$ *vertices and* $|E| = O(n^3 \log^2_\delta(\frac{n|L|}{r}))$ *edges.*

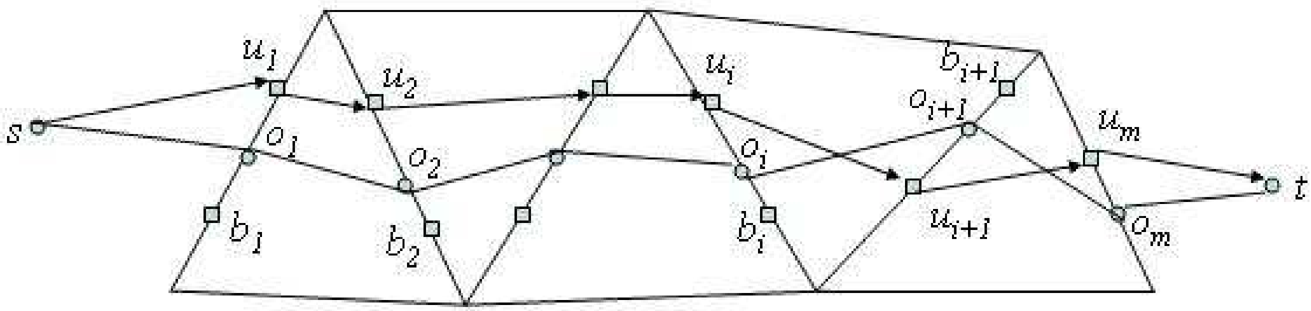

Figure 3: *Approximation of SDP*

5 Proof of Approximation Factor

Let $\pi(s,t) = [s, o_1, o_2, \ldots, o_m, t]$ be the shortest monotone descent path and it passes through the edge sequence $e_1, e_2, \ldots, e_m$. Then in each edge e_i the path $\pi(s,t)$ must pass through a pair of Steiner-points which are closest to o_i, say u_i and b_i (See Figure 3) such that $z(u_i) \geq z(o_i) \geq z(b_i)$. Note that, in degenerate case point u_i may be o_i and o_i may coincide with b_i. Let us consider the path $\pi^*(s,t) = [s, u_1, u_2, \ldots, u_m, t]$.

Lemma 1 $\pi^*(s,t)$ *is monotone descent.*

To prove the above Lemma we need to prove the following statement.

Lemma 2 *Let p and q be two points on two consecutive members e_i and e_{i+1} of $\mathcal{E}$ which bounds a face f, and $z(p) = z(q)$. Now, if a line ℓ on face f intersects both e_i and e_{i+1}, and is parallel to the line segment $[p,q]$, then all the points on ℓ have the same z-coordinate.*

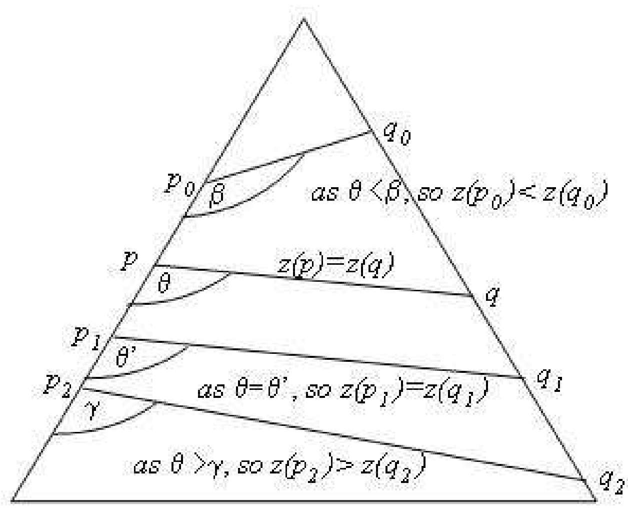

Figure 4: *Illustration of Lemma 2 and Corollary 3*

Proof. Consider a horizontal plane h at altitude $z(p)$. The intersection of the face f and the plane h is the line segment $[p, q]$. Consider another horizontal plane h' through a point r on line ℓ. The intersection of f and h' must be line parallel to $[p, q]$, and hence it coincides with the line ℓ. Thus, all the points on the line ℓ have the same z-coordinate. $\square$

From the above Lemma, we get the following Corollary.

Corollary 3 *If $[p, q]$ makes an angle θ with e_i (in anticlockwise direction see Figure 4) and $[v, w]$ ($v \in e_i$ and $w \in e_{i+1}$) be the line segment that makes an angle β with e_i (in anticlockwise direction). Now*

1. *$\theta = \beta$ iff $z(v) = z(w)$,*
2. *$\theta > \beta$ iff $z(v) > z(w)$, and*
3. *$\theta < \beta$ iff $z(v) < z(w)$.*

Now we have to prove that the path $\pi^*(s, t)$ is monotone descent. Without loss of generality we can say that $z(s) \geq z(u_1)$. We will prove this using contradiction. So the path from s to u_1 is monotone descent. Let us consider that path s to u_{k-1} along $\pi'(s, t)$ is monotone descent and the path segment u_{k-1} to u_k is not monotone descent. Then $z(u_{k-1}) < z(u_k)$. Again the path segment o_{k-1} to o_k is monotone descent and $z(u_k) > z(o_k)$ (As $z(u_k) > z(b_k)$). So there is an Isohypse-Steiner-point $du_k \in [u_k, o_k]$ (by Lemma 2 and Corollary 3, see Figure 5) such that path from s to du_k is monotone descent. This contradicts the assumption that u_k is closet to o_k.

From the above discussion, we can conclude that the path $\pi^*(s, t)$ is monotone descent. Hence Lemma 1 follows.

We will need the following definitions of [3] to prove the approximation factor.

Let s_i be a segment of $\pi(s, t)$ crossing face f_i and C_v be sphere around v. Each s_i, must be of one of the following types:

(i) $s_i \cap C_v = \emptyset$, ii) $s_i \cap C_v$=subsegment of s_i, and iii) $s_i \cap C_v = s_i$.

Let $C_{\sigma_1}, C_{\sigma_2}, \ldots, C_{\sigma_k}$ be a sequence of spheres (listed in order from s to t) intersected by type ii) segments of $\pi(s, t)$ such that $C_{\sigma_j} \neq C_{\sigma_{j+1}}$. Now define subpaths of $\pi(s, t)$ as being of two kinds:

Definition 1 *Between-sphere subpath: A path consisting of a type ii) segment followed by zero or more consecutive type i) segments followed by a type ii) segment. These subpaths will be denoted as $\pi(\sigma_j, \sigma_{j+1})$ whose first and last segments intersect C_{σ_j} and $C_{\sigma_{j+1}}$, respectively. We will also consider paths that begin or/and end at a vertex to be a degenerate case of this type of path containing only type i) segments.*

Definition 2 *Inside-sphere subpath: A path consisting of one or more consecutive type iii) segments all lying within same C_{σ_j}; these are denoted as $\pi(\sigma_j)$.*

Lemma 4 *Length of the any Inside-sphere subpath $|\pi(\sigma_j)|$ is at most $2\epsilon h$, where h is maximum among all h_v.*

Proof. Let us consider any face f inside a sphere C_v. The subtriangular face $u'vw'$ of f has two side of length $\frac{\epsilon h_v}{n}$. So the length of any segment s_i within $u'vw'$ can be at most $2\epsilon h_v$ (by triangular inequality). There may be at most n such subtriangles inside C_v. Hence the lemma follows. $\square$

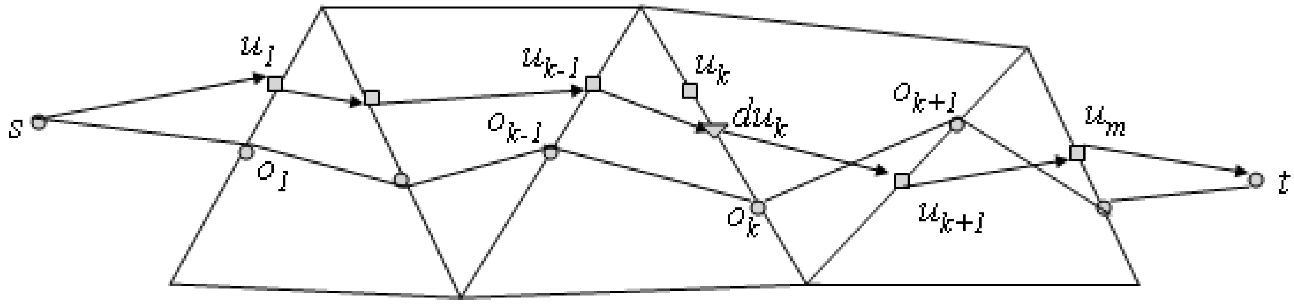

Figure 5: *Proof of Lemma 1*

Lemma 5 *If $\pi(s,t)$ consists of only type i) segments then $|\pi^*(s,t)| \leq (1+2\epsilon)|\pi(s,t)|$, where $|\pi(a,b)|$ implies the length of the path $\pi(a,b)$.*

Proof. We have

$$\begin{aligned}
|\pi^*(s,t)| &= |su_1| + |u_1u_2| + |u_2u_3| + \ldots + \\
&\quad +|u_{m-1}u_m| + |u_mt| \\
&\leq |so_1| + |o_1u_1| + |u_1o_1| + |o_1o_2| + \\
&\quad +|o_2u_2| + |u_2o_2| + |o_2o_3| + \\
&\quad +|o_3u_3| + |u_3o_3| + \ldots + |o_{m-1}u_{m-1}| \\
&\quad +|o_{m-1}o_m| + |o_mu_m| + |u_mo_m| + |o_mt| \\
&= |so_1| + |o_1o_2| + |o_2o_3| + \ldots + \\
&\quad +|o_{m-1}o_m| + |o_mt| + \\
&\quad +2\{|o_1u_1| + |o_2u_2| + \ldots + |u_mo_m|\} \\
&\leq |so_1| + |o_1o_2| + |o_2o_3| + \ldots + \\
&\quad +|o_{m-1}o_m| + |o_mt| + \\
&\quad +2\epsilon\{|o_1o_2| + |o_2o_3| + \ldots + |o_{m-1}o_m|\} \\
&\quad \text{(using Claim 1, Lemma 2 and Corollary 3)} \\
&\leq (1+2\epsilon)\pi(s,t).
\end{aligned}$$

□

From the property 1, Lemma 4 and Lemma 5 we can conclude that there exists a path $\pi^*(s,t)$ such that $|\pi^*(s,t)| = (1+\epsilon)|\pi(s,t)|$. Dijkstra's algorithm may output a monotone descent path $\pi^{**}(s,t)$ which may be a different path from $\pi^*(s,t)$. As Dijkstra's algorithm outputs a path of shortest length in graph G, we have $|\pi^*(s,t)| \geq |\pi^{**}(s,t)|$. So, we can conclude the following theorem.

Theorem 6 *Let $0 < \epsilon < \frac{1}{2}$. Let $DFR(s)$ be a terrain with n faces and let s and t be two of its vertices. An approximation $\pi'(s,t)$ of a shortest monotone path $\pi(s,t)$ can be computed such that*

$\pi'(s,t) \leq (1+\epsilon)\pi(s,t)$. *The approximate path can be computed in* $O(|V|\log|V|+|E|)$ *time using Dijkstra's algorithm or in* $O(|V|\log|V|)$ *time using Bushwhack algorithm, where* $|V| = O(n^2 \log_\delta(\frac{n|L|}{r}))$ *and* $|E| = O(n^3 \log_\delta^2(\frac{n|L|}{r}))$. *Here,* $\delta = (1+\epsilon sin\theta)$ *(by virtue of Claim 3), where* θ, L, r *denote the smallest angle among the triangles, the length of the longest edge, and the length of smallest* r_v*, respectively.*

Proof. Property 1 says that the length of the subpaths between any two spheres C_{v_a} and C_{v_b} is at least $(1-2\epsilon)h$, where h is maximum among all h_v. Lemma 4 says that length of the any Inside-sphere subpath $|\pi(\sigma_j)|$ is at most $2\epsilon h$. Lemma 5 says that the approximation factor for the paths that does not contains Inside-sphere is $\pi(s,t) = (1+2\epsilon)\pi'(s,t)$. Combining Property 1, Lemma 4 and Lemma 5 we can conclude that the approximation factor for the Between-spheres path is $\pi(s,t) = \frac{2\epsilon}{(1-2\epsilon)}\pi'(s,t)$. Hence the lemma follows. □

In Aleksandrov et al. [4] it is shown that by placing Steiner points in the interior of triangles (i.e., on angular bisectors), shortest paths on polyhedral surfaces can be approximated within a factor of $(1+\epsilon)$, while placing a reduced number of Steiner points as compared to that in [3]. The reduction is by a factor of $\frac{1}{\sqrt{\epsilon}}$. It turns out that we can replace the set of Steiner points as introduced in Section 4.1 in this paper by that on bisectors as in [4] and then insert the corresponding Isohypse points on the bisectors. The distance between any two Steiner points belonging to neighboring bisectors is defined to be the length of the local shortest descending path restricted to lie within the corresponding triangles. It turns out that Theorem 6 is still valid and we will achieve a reduction in the number of Steiner points by approximately a factor of $\frac{1}{\sqrt{\epsilon}}$.

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