A note on drawing direction-constrained paths in 3D

Ethan Kim* Giuseppe Liotta† Sue Whitesides*

Abstract

We study the problem of drawing a graph-theoretic path where each edge is assigned an axis-parallel direction in 3D. Di Battista *et al.*[3] gives a combinatorial characterization for such path drawings that start at the origin and reach a point in an octant. In this paper, we consider the drawability question for such paths that start at the origin and reach a point in a quadrant or an axis. We show that neither the necessity nor the sufficiency of the characterization given in [3] extends immediately to handle these cases. Furthermore, we give necessary conditions for such reachability, and also give examples to show they are not sufficient.

1 Introduction

Orthogonal drawings arise in applications in diverse fields such as information visualization and VLSI circuit layout. One of the most successful methodologies for generating 2D orthogonal layouts of graphs is the so-called Topology-Shape-Metrics approach, where the task of defining the shape of the drawing is separated from the task of determining the geometric coordinates of the vertices in the final drawing.

In contrast to its 2D counterpart, however, the Topology-Shape-Metrics approach has not been much exploited in 3D. The first step toward achieving this goal is due to Di Battista *et al.* [2, 3], who gave combinatorial characterizations of paths and cycles with given shape such that they admit simple orthogonal 3D drawings. In particular, reference [3] answers the path reachability problem: given a point p lying in an open octant in R^3 and a graph-theoretic path with an axis-parallel direction label on each edge, can we compute a drawing of the path such that the drawing starts at the origin and ends at point p while respecting the directions labels? While a combinatorial characterization is given for paths that admit such drawings, this characterization does not handle cases where p is located in a quadrant or an axis, i.e., where 2 or 1 coordinates of p are zero. In this paper, we study the path reachability problem where the final vertex is to be drawn in a quadrant or an axis, and show that the characterization given in [3] does not immediately extend to handle these cases. Indeed the necessary condition does not extend; this paper gives a necessary condition, together with examples that show it is not sufficient, and presents chracterization of low dimensional reachability as a challenging open problem.

*School of Computer Science, McGill University, Canada, {ethan, sue}@cs.mcgill.ca

†Dipartimento di Ingegneria Elettronica e dell'Informazione, Università di Perugia, Italy, liotta@diei.unipg.it

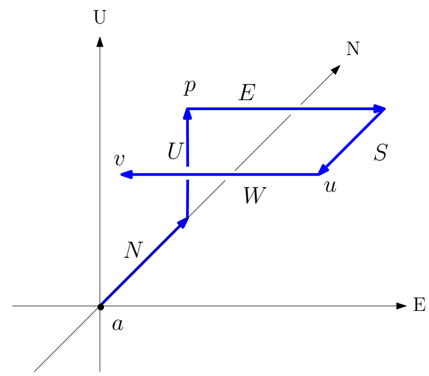

Figure 1: A drawing of $\sigma = NUESW$ that reaches the UN-quadrant

2 Preliminaries

We define a *shape path* to be a graph-theoretic path σ, where each edge is labeled with an axis-parallel direction in 3D: N, S, E, W, U, or D. For obvious reasons, we assume that the labels on adjacent edges must be perpendicular to each other. Furthermore, we often let X(X'), Y(Y'), or Z(Z') denote an axis-parallel (and its opposite) direction without referring to a specific direction. When we refer to a particular element, i.e., an edge, in a shape path σ, we often denote it by σ_i, where i is an index for the edges of σ. Also, when we refer to a subpath from vertex a to vertex b, we often denote it by σ_{ab}. We typically describe a shape path σ by giving the sequence of the direction labels on its edges in the order that they appear in σ.

A *flat* of a shape path σ is a consecutive subsequence of σ that is maximal with respect to the property that its labels are coplanar. We say a flat is *heavy* if it contains 3 or more edges, and *light* otherwise. Observe that the first edge of a flat is the last edge of the previous flat, and the last edge of a flat is the first edge of the next flat. We call such edges that belong to two consecutive flats *transition elements* (See [3].).

We say a drawing of a shape path is *simple and orthogonal* if no two edges intersect except at a shared endpoint, and every edge is drawn parallel to an axis. We let $\Gamma(\sigma)$ denote a simple orthogonal drawing of σ that respects the direction labels on the edges. Furthermore, we often use $\overline{uv}$ to denote the line segment that represents an edge uv in $\Gamma(\sigma)$. See Figure 1 for an example of a simple orthogonal drawing of a shape path. We say a drawing Γ is in *general position* if, for every pair u, v of vertices that share a coordinate, either they belong to the same flat, or one belongs to the first flat and the other to the last flat. The following lemma allows us to only consider drawings in general position.

Lemma 1 [2](General Position Lemma) *Any simple orthogonal drawing $\Gamma(\sigma)$ can be perturbed to lie in general position.*

Let τ be a not necessarily consecutive subsequence of σ. We say τ is a *canonical sequence* if (1) the labels in τ are pairwise distinct; (2) no flat of σ contains more than three labels of τ; and (3) if a flat F of σ contains two or more labels of τ, then $\tau \cap F$ is a consecutive subsequence of σ. We often call the elements of a canonical sequence *canonical labels*, and use special notation on these labels such as $\bar{}, \hat{}, \tilde{}$ to refer to a particular occurrence of its underlying label. Moreover, we sometimes disregard the ordering of the labels in a canonical sequence by using the set notation $\{\}$. For example, $\tau = \{\bar{U}, \bar{N}, \bar{E}\}$ completely describes a canonical sequence. When we consider canonical sequences in this fashion, we can thus use any conventional set operation on them.

Finally, we define the *type* of a canonical sequence to be the set of its direction labels by disregarding the ordering of the canonical labels in the sequence. For example, $\tau = \{\bar{E}, \bar{N}, \bar{U}\}$ is a canonical sequence of type $\{U, N, E\}$.

Related Work Di Battista *et al.* characterizes the path reachability problem for cases where the final vertex is to be drawn in an octant as follows.

Theorem 2 [3] *Let σ be a shape path and let p be a point in the XYZ octant. Then there exists a simple orthogonal drawing of σ that starts at the origin and terminates at p if and only if σ contains a canonical sequence of type $\{X, Y, Z\}$.*

The next step towards extending the Topology-Shape-Metrics approach to 3D is taken by the same authors [1, 2], who give a characterization for cycles that admit simple orthogonal drawings. In particular, the following results provide useful tools for our discussion.

Lemma 3 [1] *Let $\Gamma(\sigma)$ be a drawing of a 2D shape path σ that starts at the origin, and that has the last edge uv intersecting the X-axis orthogonally. Let Y, Y' be the direction labels that denote the direction of uv and its opposite direction. Then σ contains a canonical sequence of type $\{X, Y, Y'\}$.*

Lemma 4 [2] *Let τ_1 and τ_2 be two canonical sequences for* a shape path *σ such that (1) τ_1 and τ_2 have no direction labels in common, and (2) no flat of σ contains an element from τ_1 and an element from τ_2. Then $\tau_1 \cup \tau_2$ is canonical for σ.*

Lemma 5 [1] *Let $\sigma = \sigma_1 \dots \sigma_i \sigma_{i+1} \dots \sigma_n$ be* a shape path *such that there is a light flat $F_l = \sigma_i \sigma_{i+1}$. If τ_1 and τ_2 are two canonical sequences for σ such that $\tau_1 \subseteq \sigma_1 \dots \sigma_i$ and $\tau_2 \subseteq \sigma_{i+1} \dots \sigma_n$, and τ_1 and τ_2 have no direction labels in common, then $\tau_1 \cup \tau_2$ is canonical for σ.*

On another note, the characterization given for 3D orthogonal shape cycles in [2] does not extend to even seemingly simple graphs. Di Giacomo *et al.* [4] discovered a shape graph, consisting of only 3 cycles, such that every simple shape cycle induced by its vertices admits a simple orthogonal drawing, but the shape graph itself cannot be drawn in a simple orthogonal manner.

Finally, the latest result on orthogonal shape graphs is by Patrignani [6], where it is shown that the 3D drawability of shape graphs is, in general, NP-hard.

3 Reaching a quadrant orthogonally

In this section, we discuss the properties of shape path drawings such that the first vertex is at the origin, and the last edge reaches a quadrant orthogonally. We say an edge uv *orthogonally intersects the XY-quadrant* if uv is perpendicular to the XY-quadrant, and an endpoint of uv lies in the XY-quadrant, or u and v lie in the XYZ and XYZ' octants respectively. Similarly, we say an edge uv *orthogonally intersects the X-axis* if uv is perpendicular to the X-axis, and an endpoint of uv lies in the X-axis, or u and v lie in two distinct quadrants adjacent to the X-axis. The following natural generalization of Lemma 3 appears in [2].

Lemma 6 [2] *Let $\Gamma(\sigma)$ be a drawing of a 3D shape path σ that starts at the origin, and that has its last edge uv orthogonally intersecting the XY-quadrant. Then, σ contains a canonical sequence of type $\{X, Y, Z, Z'\}$.*

Here we show that the above proposition does not hold by exhibiting a counterexample[1].

Counterexample 7 *The shape path $\sigma = NUESW$ does not contain a canonical sequence of type*

[1] We note that the changes made from Lemma 6 to Lemma 8 does not affect the validity of the characterization for 3D orthogonal cycle drawings given in [2], as verified in [5].

$\{N, U, E, W\}$, yet σ admits a drawing that starts at the origin, and has its last edge orthogonally intersecting the UN quadrant as shown in Figure 1.

Hence, we repair Lemma 6 as follows.

Lemma 8 *Let $\Gamma(\sigma)$ be a drawing of a 3D shape path σ that starts at the origin, and has its last edge uv orthogonally intersecting the XY-quadrant. Then σ has a canonical sequence that contains $\{X, Y, Z, Z'\}$ as a subset.*

Proof Sketch: We use induction on the length of σ, denoted by $|\sigma|$. When $|\sigma| = 4$, the statement holds trivially. Suppose the statement holds for all values $4 \leq |\sigma| \leq k$, and consider the case where $|\sigma| = k + 1$. If there exists an edge other than uv that orthogonally intersects the XY-quadrant, we are done by the induction hypothesis. So, we assume no other edge intersects the XY quadrant orthogonally.

Let F_{pv} denote the flat that contains uv, and consider the first vertex p of F_{pv}. We assume, without loss, that u and v belong to two distinct octants adjacent to the XY quadrant (i.e. uv crosses the XY quadrant.). Otherwise, we can stretch the edge uv slightly to cross the quadrant without otherwise changing the drawing. Then, we assume, without loss of generality, that u is in the UNE octant, v is in the UNW octant, and thus uv is crossing the UN quadrant. Further, we assume F_{pv} is a NSEW flat. Other cases can be handled by applying the same arguments after permuting the direction labels. Then, we distinguish 9 different cases depending on the location of p: p may belong to one of the four octants UNE, UNW, USE, USW, or one of the four quadrants UN, UE, US, UW, or the U axis. Here we discuss one of these cases, which covers Counterexample 7, as a typical example.

p **lies in the UN-quadrant:** Let a denote the first vertex of σ. By the general position assumption, a and p must lie in the same flat. Since p is the first vertex of the flat containing $\overline{uv}$, we have that $\overline{pp'}$ is a transition element, and there are two cases: (i) There is a flat F_{ap} that starts at a and ends at p. Then there is a light flat between F_{ap} and F_{pv}; (ii) There is a flat $F_{ap'}$ that starts at a and ends at p'. By Theorem 2, σ_{ap} contains $\tau_1 = \{\bar{U}, \bar{N}\}$. Depending on where $\overline{uv}$ intersects the UN quadrant, there are two cases:

1. p lies south of $\overline{uv}$: By Lemma 3, σ_{pv} contains $\tau_2 = \{\hat{E}, \hat{N}, \hat{W}\}$. In case (i), $\{\bar{U}\} \cup \tau_2$ is canonical by Lemma 5. In case (ii), then $\overline{pp'}$ is either N or S. Thus, $\overline{pp'}$ is neither $\hat{E}$, nor $\hat{W}$. Furthermore, since τ_2 belongs to the flat F_{pv}, $\hat{N}$ must appear between $\hat{E}$ and $\hat{W}$, so $\overline{pp'}$ cannot be $\hat{N}$. Hence, $\overline{pp'} \notin \tau_2$. By Lemma 4, $\{\bar{U}\} \cup \tau_2$ is canonical.

2. p lies north of $\overline{uv}$: By Lemma 3, σ_{pv} contains $\tau_3 = \{\tilde{E}, \tilde{S}, \tilde{W}\}$. In case (i), then $\tau_1 \cup \tau_3$ is canonical by Lemma 5. In case (ii), then $\overline{pp'}$ is either N or S. Thus, $\overline{pp'}$ is neither $\tilde{E}$ nor $\tilde{W}$. Furthermore, since τ_3 belongs to the flat F_{pv} it follows that $\tilde{S}$ must appear between $\tilde{E}$ and $\tilde{W}$, so $\overline{pp'}$ cannot be $\tilde{S}$. Hence, $\overline{pp'} \notin \tau_3$. By Lemma 4, $\tau_1 \cup \tau_3$ is canonical. □

4 Necessity

In this section, we give necessary conditions for shape paths that start at the origin and end at a point on a quadrant or an axis. We first consider shape paths that are drawn to reach an axis orthogonally.

Lemma 9 *Let $\Gamma(\sigma)$ be a drawing of a 3D shape path σ that starts at the origin, and has its last edge uv orthogonally intersecting the X-axis. Then, σ has a canonical sequence that contains $\{X, Y, Y', Z, Z'\}$ as a subset.*

Proof Sketch: We use induction on the length of σ. Then, as in the proof of Lemma 8, we may assume that no other edge in σ may orthogonally intersect the X-axis. Assume, without loss, that u belongs to the UE-quadrant, and v belongs to the UW-quadrant. Then $\overline{uv}$ crosses the U-axis. Now, consider the flat F_{pv} that contains uv, and let p denote the first vertex of F_{pv}. We assume, without loss, that F_{pv} is an NSEW flat. Then there are 8 cases to consider, depending on the position of p in the drawing: p may belong to one of the 4 octants UNE, UNW, USE, USW, or one of the 4 quadrants UN, UE, US, UW. We can show, for each case, that σ has a canonical sequence that contains $\{X, Y, Y', Z, Z'\}$ as a subset. □

Notice that this result is analogous to Lemma 8. Similarly to Lemma 8, Lemma 9 cannot be strengthened so that σ contains exactly those 5 labels. Consider the following example.

Example 10 *The shape path $\sigma = NUSEDW$ can be drawn so that $\Gamma(\sigma)$ starts at the origin and intersects the U axis orthogonally. See Figure 2. However, σ does not contain a canonical sequence of type $\{U, N, E, S, W\}$.*

Note that the statement of Lemma 9 requires that the last edge of σ orthogonally intersects the X axis. Now we show that the same necessary condition holds for shape path drawings without this requirement.

Theorem 11 *Let $\Gamma(\sigma)$ be a drawing of a 3D shape path σ that starts at the origin, and has ends at some point p in the X axis. Then, σ has a canonical sequence that contains $\{X, Y, Y', Z, Z'\}$ as a subset.*

Proof. Let a be the first vertex of σ, and let uv the last edge of σ. By assumption, v lies in the X axis. If uv is orthogonal to the X axis, we are done by Lemma 9.

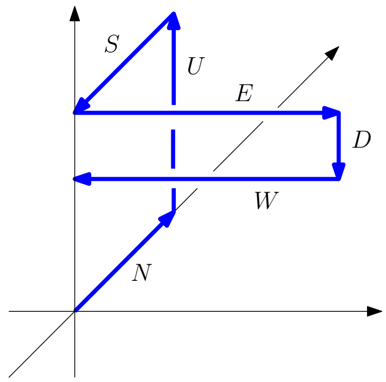

Figure 2: A drawing of a shape path $\sigma = NUSEDW$ that intersects the U axis orthogonally; but σ itself does not contain a canonical sequence of type $\{U, N, E, S, W\}$.

Otherwise, because σ is a 3D shape path, there must exist an edge bc before uv such that $\overline{bc}$ touches the axis orthogonally, and σ_{ac} is a 3D shape path. Then, σ_{ac} contains the desired canonical sequence by Lemma 9. □

Similarly, the same necessary condition in Lemma 8 holds for shape path drawings that start at the origin, and terminate at some point in a quadrant.

Theorem 12 *Let $\Gamma(\sigma)$ be a drawing of a 3D shape path σ that starts at the origin, and ends at some point in the XY quadrant. Then, σ has a canonical sequence that contains $\{X, Y, Z, Z'\}$ as a subset.*

Proof. Let a denote the first vertex of σ, and let uv be the last edge of σ. If $\overline{uv}$ is orthogonal to the XY-quadrant, we are done by Lemma 8. By the same reasoning, if there exists an edge that, in the drawing $\Gamma(\sigma)$, intersects the XY-quadrant orthogonally, we are done. Otherwise, $\Gamma(\sigma)$ must reach the XY-quadrant either through the X-axis or through the Y-axis. Let pq denote the last edge of σ that enters the XY-quadrant (i.e. q lies in the XY-quadrant while p does not.). Then, pq must intersect either the X-axis or the Y-axis orthogonally. Furthermore σ_{aq} is a 3D path. If pq intersects the X-axis, by Lemma 9, there is a canonical sequence in σ_{aq} that contains $\{X, Y, Y', Z, Z'\}$ as a subset. If pq intersects the Y-axis, by Lemma 9, there is a canonical sequence in σ_{aq} that contains $\{X, X', Y, Z, Z'\}$ as a subset. □

5 Insufficiency

Observe that Lemmas 11 and 12 may serve as necessary conditions for characterizing shape paths that start at the origin and end at some point in a quadrant or an axis. If the converse of these two lemmas were true, we would have characterizations of such shape paths that can be checked in linear time, using the algorithm of Di Battista *et al.* [2, 3]. Unfortunately, the converse of these two lemmas does not hold. We offer the following examples.

Example 13 *Let σ =UNESDW. Then, $\tau = UNESW$ is a canonical sequence for σ. However, σ does not admit a simple orthogonal drawing Γ that starts at the origin, and terminates at some point in the U-axis.*

Example 14 *Let σ =NEDSWU. Then $\tau = NEWU$ is a canonical sequence for σ. However, σ does not admit a simple orthogonal drawing Γ that starts at the origin, and terminates at some point in the UN-quadrant.*

6 Conclusion & open problems

In this note, we have shown that the characterization of simple orthogonal shape paths given in [3] does not extend immediately to shape paths reaching a point in a quadrant or an axis. We have repaired a lemma of [2], and used this to provide necessary conditions for reachability of axes and quadrants. An obvious question, then, would be to provide a characterization for such reachability. More generally, it would be of interest to characterize d-dimensional shape paths reaching a k-dimensional subspace, as [3] provides a characterization only for reaching $k = d$-dimensional subspaces.

References

[1] Giuseppe Di Battista, Ethan Kim, Giuseppe Liotta, Anna Lubiw, and Sue Whitesides. Orthogonal drawings of cycles in 3D space. *In preparation.*

[2] Giuseppe Di Battista, Giuseppe Liotta, Anna Lubiw, and Sue Whitesides. Orthogonal drawings of cycles in 3D space. In *8th Intl. Sympos. on Graph Drawing*, pages 272–283, 2001.

[3] Giuseppe Di Battista, Giuseppe Liotta, Anna Lubiw, and Sue Whitesides. Embedding problems for paths with direction constrained edges. *Theor. Comput. Sci.*, 289(2):897–917, 2002.

[4] Emilio Di Giacomo, Giuseppe Liotta, and Maurizio Patrignani. A note on 3D orthogonal drawings with direction constrained edges. *Inf. Process. Lett.*, 90(2):97–101, 2004.

[5] Ethan D. H. Kim. Three-dimensional orthogonal graph drawing with direction constrained edges. Master's thesis, School of Computer Science, McGill University, 2007.

[6] Maurizio Patrignani. Complexity results for three-dimensional orthogonal graph drawing. In *13th Intl. Sympos. on Graph Drawing*, pages 368–379, 2005.