Restricted Edge Contractions in Triangulations of the Sphere with Boundary

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Abstract

Given a surface triangulation T of and a subset X of its vertex set $V(T)$, we define a restricted edge contraction as a contraction of an edge connecting X and $V(T)-X$. Boundary vertices in $V(T)-X$ are only allowed to be contracted to the boundary vertices in X adjacent through boundary edges. In this paper, we prove that if a triangulation T of the sphere with boundary satisfies some connectivity conditions, then all the vertices in $V(T)-X$ can be merged into X by restricted edge contractions. We also prove that the similar properties hold for a triangulation of the sphere with features.

1 Introduction

A *triangulation* of a closed surface S (connected, compact 2-manifold without boundary) is a simple graph embedded on S, such that each face is homeomorphic to a 2-cell and is bounded by three edges, and any two faces share at most one edge. For a triangulation T, we denote the vertex set, edge set and face set by $V(T)$, $E(T)$ and $F(T)$, respectively. We refer to [6] for embeddings of graphs into surfaces and to [9] for triangulations in topological graph theory.

Given a surface triangulation T and a subset X of its vertex set $V(T)$, we define a restricted edge contraction as a contraction of an edge connecting X and $V(T)-X$. Boundary vertices in $V(T)-X$ are only allowed to be contracted to the boundary vertices in X adjacent through boundary edges. We consider restricted edge contractions because it can be applied to enforce topology preservation in vertex-based region growing algorithms on surface triangulations [5].

In [7] we proved that if T is a triangulation of the sphere and $X \subset V(T)$ is a seed vertex set satisfying $|X| \geq 4$ then all vertices in $V(T)-X$ can be merged into X by a sequence of restricted edge contractions. This says that, for any triangulation of the sphere, if a seed vertex set contains a sufficient number of vertices, then all vertices which do not belong to the seed vertex set can be merged into seed vertices by a sequence of restricted edge contractions regardless of the configuration of the seed vertices. We also showed that this property does not hold for closed surfaces other than the sphere [7]. In this paper, we consider whether the property holds for the sphere with boundary.

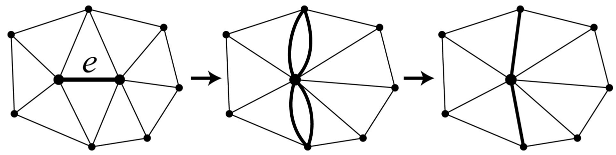

Figure 1: Edge contraction of an edge e.

2 Edge Contractions

Contraction of an edge e is the operation that consists of deleting e, identifying the endpoints of e, and replacing the two resulting digons (2-sided faces) by two single edges, as shown in Figure 1. If the contraction of an edge in a triangulation T results in another triangulation of the surface on which T is embedded, the edge is said to be *contractible*. Non-contractible edges are not allowed to be contracted.

An edge e in a triangulation T of a closed surface is contractible if and only if e satisfies the following conditions:

- e does not lie on any non-facial triangle.
- T is not K_4 embedded on the sphere (i.e. a tetrahedron).

A *non-facial triangle* is a 3-cycle that does not bound a face.

A triangulation is said to be *irreducible* if it has no contractible edges. Steinitz [10] showed that the only irreducible triangulations of the sphere is K_4. Barnette and Edelson [1] proved that there are only finitely many irreducible triangulations for every closed surface. Nakamoto and Ota [8] showed that the maximum number of vertices in irreducible triangulations of a surface S has a linear bound with respect to the genus of S.

A triangulation T of a surface *with boundary* must satisfy the additional requirement: for each boundary component B_i of the surface, there is a cycle in T which coincides with B_i. The edges of these cycles have only one incident face and are called *boundary edges*. The other edges are incident to two faces and are called *interior edges*. A vertex is called a *boundary vertex* if it

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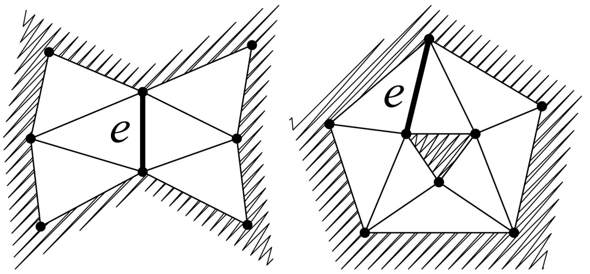

Figure 2: The endpoints of a bridge e lie either on the same boundary component (left) or on distinct boundary components (right).

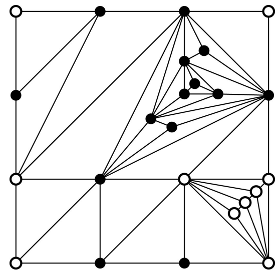

Figure 3: A triangulation of the torus (black dots: seed vertices; white dots: free vertices). All edges connecting a seed vertex and a free vertex is non-contractible.

is incident to at least one boundary edge; otherwise it is called an *interior vertex*. We denote the boundary components of T by $B_1(T), \ldots, B_k(T)$ (k is the number of the boundary components of T). The vertex set and edge set of $B_i(T)$ is denoted $V(B_i(T))$ and $E(B_i(T))$, respectively.

An edge is called a *bridge*, if its endpoints are boundary vertices but it is not a boundary edge. A bridge may connect distinct boundary components (Figure 2). An edge e in a triangulation T of a surface with boundary is contractible if and only if e satisfies the following conditions:

- e does not lie on any non-facial triangle.
- e is not a bridge.
- T is not K_3 embedded on the sphere with one boundary component (i.e. a triangle).

Edge contraction in simplicial 2-complexes homeomorphic to a surface is essentially equivalent to edge contraction in surface triangulations. Simplicial 2-complexes homeomorphic to a surface with or without boundary [4], simplicial 2-complexes homeomorphic to a surface with features [3], general simplicial 2- and 3-complexes [2], and general simplicial 2-complexes with features [11] are treated in the literature on mesh simplification.

3 Restricted Edge Contractions in triangulations of a closed surface

Given a triangulation T of a closed surface and a subset X of its vertex set $V(T)$, which is called a *seed vertex set*, we define a *restricted edge contraction* as a contraction of an edge connecting X and $V(T) - X$. Vertices in X are called *seed vertices* and the other vertices are called *free vertices*. Restricted edge contraction of an edge $e = uv$ (where u is a seed vertex and v is a free vertex) merges the free vertex v into the seed vertex u. Thus, during restricted edge contractions, seed vertices remain unchanged and free vertices are merged into seed vertices.

Let T be a triangulation of a closed surface and $X \subset V(T)$ be a seed vertex set. A triangulation T' is called a *seed triangulation* or an *X-triangulation*, if the vertex set of T' is equal to X. If a seed triangulation can be obtained from T by a sequence of restricted edge contractions, then we say that T is contractible to a seed triangulation or to an X-triangulation. We proved the following results in [7].

Proposition 1 *Let T be a triangulation of the sphere and $X \subset V(T)$ be a seed vertex set. If $|X| \geq 4$, then T is contractible to an X-triangulation by any sequence of restricted edge contractions.*

Note that we only contract contractible edges.

Proposition 2 *For every closed surface other than the sphere, there exists a triangulation T and a seed vertex set $X \subset V(T)$ such that T is not contractible to any X-triangulation.*

Figure 3 shows such an example for the torus (each pair of parallel sides of the rectangle should be identified).

4 Restricted Edge Contractions in triangulations of the sphere with boundary

For restricted edge contractions in triangulations of a surface with boundary, a free vertex on a boundary component B_i should be merged into an adjacent seed vertex on B_i by contracting the boundary edge connecting them. Because of this, the candidate edges for restricted edge contractions do not contain bridges.

The sphere has the property that for any triangulation T if a seed vertex set X has a sufficient number of vertices then T is contractible to an X-triangulation regardless of the configuration of the seed vertices. In this section we consider whether the property holds for the sphere with boundary. Unfortunately the property does not hold for the sphere with boundary. There exists a

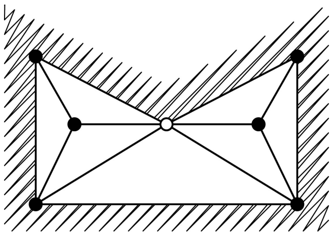

Figure 4: A triangulation which is not contractible to any seed triangulation (black dots: seed vertices; white dots: free vertices).

triangulation of the sphere with boundary which is not contractible to any seed triangulation for some configuration of the seed vertices. Figure 4 illustrates an example which is not contractible to any seed triangulation. However, we can prove that if a triangulation T of the sphere with boundary satisfies some connectivity conditions and a seed vertex set X has a sufficient number of vertices then T is contractible to an X-triangulation regardless of the configuration of the seed vertices. To prove this, we need some lemmas.

Lemma 3 *Let T be a triangulation of the sphere with boundary and $X \subset V(T)$ be a seed vertex set. If all the vertices on the boundary of T are contained in X, then T is contractible to an X-triangulation by any sequence of restricted edge contractions.*

Proof. We create a triangulation of the sphere from T by filling every boundary component of T. For each boundary component B_i of T, we add a dummy seed vertex d_i, edges $\{e = d_i v \mid v \in V(B_i)\}$ and faces $\{f = d_i uv \mid uv \in E(B_i)\}$ to B_i. After filling all the boundary components, we get a triangulation T' of the sphere. Contractibility of an edge connecting a seed vertex and a free vertex of T is the same as that of the corresponding edge in T'. Since T' is contractible to a seed triangulation by a sequence of restricted edge contractions, T can be contracted to an X-triangulation by applying corresponding restricted edge contractions. Furthermore, since any sequence of restricted edge contractions in T' leads to a seed triangulation, T can be contracted to an X-triangulation by any sequence of restricted edge contractions. □

A 2-path (u, v, w) in a triangulation T is called *non-facial* if uvw is not a face of T, and it is called *chordal* if its endvertices u and w are boundary vertices and lie on the same boundary component and its inner vertex v does not lie on the boundary component. Figure 5 illustrates a non-facial chordal 2-path.

Lemma 4 *Let T be a triangulation of the sphere with boundary and $X \subset V(T)$ be a seed vertex set. If T has neither bridges nor non-facial chordal 2-paths and $|X \cap V(B_i)| \geq 3$ for each boundary component B_i of T, then all the free vertices on the boundary of T can be contracted by any sequence of restricted edge contractions on the boundary.*

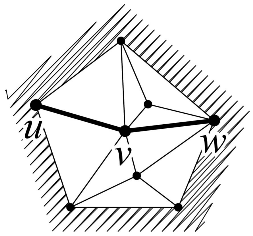

Figure 5: A non-facial chordal 2-path (u, v, w).

Proof. By assumption boundary edges cannot lie on any non-facial triangle. Thus all boundary edges connecting a seed vertex and a free vertex is contractible. Since contracting a boundary edge creates neither bridges nor non-facial chordal 2-paths, all the free vertices on the boundary of T can be contracted by any sequence of restricted edge contractions on the boundary. □

Combining Lemma 3 and Lemma 4 we get the following proposition.

Proposition 5 *Let T be a triangulation of the sphere with boundary and $X \subset V(T)$ be a seed vertex set. If T has neither bridges nor non-facial chordal 2-paths and $|X \cap V(B_i)| \geq 3$ for each boundary component B_i of T, then T is contractible to an X-triangulation.*

Proof. First we contract all the free vertices on the boundary. Lemma 4 ensures that this can always be done (by any sequence of restricted edge contractions on the boundary). Then, since all boundary vertices of the resulting triangulation are seed vertices, all the remaining free vertices can be contracted (by any sequence of restricted edge contractions) by Lemma 3. □

In the proof, we first contract all boundary free vertices and then contract all interior free vertices. If we contract interior free vertices before contracting boundary free vertices, bridges and non-facial chordal 2-paths might be created and some free vertices on the boundary could not be contracted.

When a triangulation does not satisfy the assumption of Proposition 5, we can modify the triangulation to satisfy the assumption; we remove bridges and non-facial chordal 2-paths by applying edge split to them. *Edge split* of an edge e is the operation that subdivides e into two edges and subdivides each face incident to e into two faces accordingly, as shown in Figure 6. First

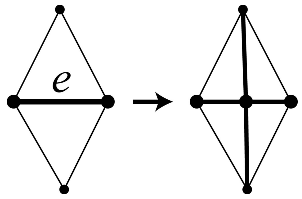

Figure 6: Edge split of an edge e.

we remove all bridges by splitting them, and then we remove all non-facial chordal 2-paths by splitting their edges. We can remove all bridges and non-facial chordal 2-paths after a finite sequence of edge splits.

5 Restricted Edge Contractions in triangulations of the sphere with features

By a surface *with features* we mean a surface with specified (possibly intersecting) curves and points on it; they are called feature curves and feature points, respectively. A triangulation T of a surface with features must satisfy the following requirements: for each feature curve, there are edges (called *feature edges*) in T which coincides with the feature curve; for each feature point, there is a vertex (called a *feature vertex*) in T which coincides with the feature point. If feature curves intersect themselves, each intersection point must have a corresponding vertex (called *corner vertex*) in T. The closure of each connected component of the feature curves minus the corner vertices is called a *feature component*. Vertices in T which correspond to endpoints of feature curves are called *anchor vertices*. Cycles consisting of feature edges in T are called *feature cycles*.

For restricted edge contractions in triangulations of a surface with features, we should include every feature vertex, corner vertex and anchor vertex in a seed vertex set, and a free vertex on a feature component F_i should be merged into an adjacent seed vertex on F_i by contracting the feature edge connecting them. For triangulations of a surface with features we modify the definition of bridges and chordal 2-paths as follows. An edge is called bridge, if its endpoints are vertices on the feature edges but it is not a feature edge. A 2-path (u, v, w) in a triangulation T is called *chordal* if its end-vertices u and w are vertices of the feature edges and lie on the same feature component and its inner vertex v does not lie on the feature component.

We can prove the following lemmas and proposition similarly as in triangulations of the sphere with boundary.

Lemma 6 *Let T be a triangulation of the sphere with features and $X \subset V(T)$ be a seed vertex set. If all the vertices on the feature edges of T are contained in X, then T is contractible to an X-triangulation by any sequence of restricted edge contractions.*

Lemma 7 *Let T be a triangulation of the sphere with features and $X \subset V(T)$ be a seed vertex set. If T has neither bridges nor non-facial chordal 2-paths and $|X \cap V(C_i)| \geq 3$ for each feature cycle C_i of T, then all the free vertices on the feature edges of T can be contracted by any sequence of restricted edge contractions on the feature edges.*

Proposition 8 *Let T be a triangulation of the sphere with features and $X \subset V(T)$ be a seed vertex set. If T has neither bridges nor non-facial chordal 2-paths and $|X \cap V(C_i)| \geq 3$ for each feature cycle C_i of T, then T is contractible to an X-triangulation.*

The extension of these lemmas and proposition to triangulations of the sphere with both boundary and features is trivial.

References

[1] Barnette, D. W. and Edelson, A. L. All 2-manifolds have finitely many minimal triangulations. *Israel Journal of Mathematics*, **67** (1): 123–128, 1989.

[2] Dey, T. K., Edelsbrunner, H., Guha, S., and Nekhayev, D. V. Topology preserving edge contraction. *Publications de l'Institut Mathématique (Beograd)*, **66** (3): 23–45, 1999.

[3] Hoppe, H. Progressive meshes. In *Proceedings of SIGGRAPH*, 99–108, 1996.

[4] Hoppe, H., DeRose, T., Duchamp, T., McDonald, J., and Stuetzle, W. Mesh optimization. In *Proceedings of SIGGRAPH*, 19–26, 1993.

[5] Klein, A., Certain, A., DeRose, A., Duchamp, T., and Stuetzle, W. Vertex-based Delaunay triangulation of meshes of arbitrary topological type. Technical Report, University of Washington, 1997.

[6] Mohar, B. and Thomassen, C. *Graphs on Surfaces*. Johns Hopkins University Press, 2001.

[7] Moriguchi, M. and Sugihara, K. Restricted edge contractions in surface triangulations. In *Proceedings of KyotoCGGT2007*, to appear, 2007.

[8] Nakamoto, A. and Ota, K. Note on irreducible triangulations of surfaces. *Journal of Graph Theory*, **20** (2): 227–233, 1995.

[9] Negami, S. Triangulations. In Gross, J. L. and Yellen, J. eds., *Handbook of Graph Theory*, CRC Press, 737–760, 2004.

[10] Steinitz, E. and Rademacher, H. *Vorlesungen über die Theorie der Polyeder: unter Einschluss der Elemente der Topologie*. Springer, Berlin, 1934.

[11] Vivodtzev, F., Bonneau, G.-P., and Letexier, P. Topology-preserving simplification of 2D nonmanifold meshes with embedded structures. *The Visual Computer*, **21** (8): 679–688, 2005.