Homothetic Triangle Contact Representations of Planar Graphs*

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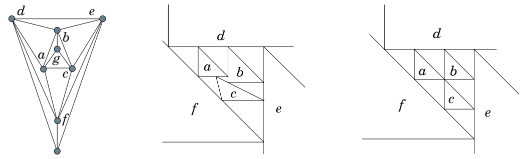
Abstract

In this paper we study the problem of computing homothetic triangle contact representations of planar graphs. Since not all planar graphs admit such a representation, we concentrate on meaningful subfamilies of planar graphs and prove that: (*i*) every two-terminal series-parallel digraph has a homothetic triangle contact representation, which can be computed in linear time; (*ii*) every partial planar 3-tree admits a homothetic triangle contact representation.

1 Introduction

Intersection graphs of geometrical objects in the plane are intensively studied both for their practical motivations and interesting theoretical properties [10, 11]. A stronger concept of intersection graph is the one of contact representation (or contact graph), where objects are not allowed to overlap. Probably the best known result about contact representations is Koebe's celebrated "Kissing lemma" stating that every planar graph has a disk contact representation [8]. De Fraysseix et al. [3] showed that every bipartite planar graph is a contact graph of vertical and horizontal segments. The computational complexity of recognizing contact graphs of segments and curves is considered in [6], and contact graphs of unit disks in [1, 5]. Contact (and intersection) graphs of translates of regular k-gons were considered in [2] where some NP-hardness results for their recognition were obtained.

A *triangle contact representation* of a graph G is a drawing such that each vertex of G is represented by a triangle and two triangles touch in a single point if their corresponding vertices are adjacent, and the triangles are disjoint otherwise. De Fraysseix et al. [4] proved that any planar graph admits a triangle contact representation. In this paper we study the problem of computing triangle contact representations of planar graphs under the restriction that the triangles are homothetic. We recall that two triangles are homothetic if they only differ in a geometric contraction or expansion.



(a) (b) (c)

Figure 1: A graph that does not admit a homothetic triangle contact representation.

Our interest in homothetic triangles is motivated by the observation by Kaufmann et al. [7] that the so called max-tolerance graphs (which have interesting applications in molecular biology [9]) are exactly the intersection graphs of homothetic triangles in the plane. Subsequently, K. Lehmann conjectured that every planar graph allows such a representation. In view of the above cited result of de Fraysseix et al. [4], it is natural to check first if every planar graph has a *contact* representation by homothetic triangles. However, one quickly sees that this is not the case. Consider, for example, the octahedron with two cubic vertices inscribed in opposite faces depicted in Figure 1(a). In any contact representation, triangles a, b, and c must be represented inside the region bounded by the representation of triangles d, e, and f, or vice versa (see Figure 1(b)). If all triangles are homothetic, then triangles a, b, c meet in one point as depicted in Figure 1(c) and it is not possible to insert triangle g.

Hence, a natural research target is to explore what families of planar graphs allow homothetic triangle contact representations, and this is the aim of our present paper. Our main results are the following:

- We prove that every *two-terminal series-parallel digraph* (*TTSP*-digraph) has a homothetic triangle contact representation, which can be computed in linear time (Section 2).
- We extend the study to larger subfamilies of planar graphs and prove that any *partial planar* 3-*tree* can be realized as a homothetic triangle contact representation (Section 3).

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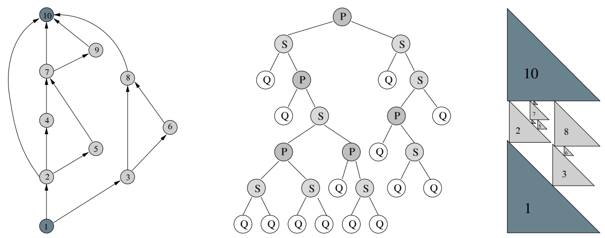


(a) (b) (c)

Figure 2: (a) A TTSP-digraph G. (b) The decomposition tree of G. (c) A strict homothetic triangle contact representation of G.

We actually prove both results in a stronger form. Namely, our triangle contact representations are such that every contact point of the representation is an inner point of a side of one of the triangles (and consequently no three triangles meet in a common point). We call such a representation a *strict* triangle contact representation.

2 Contact Representations of TTSP-digraphs

A *two terminal series-parallel digraph* (*TTSP-digraph* for short) [12] is a planar digraph that has one source and one sink, called *poles*, and it is recursively defined as follows. A single edge is a TTSP-digraph. The digraph obtained by identifying the sources and the sinks of two TTSP-digraphs is a TTSP-digraph. The digraph obtained by identifying the sink of one TTSP-digraph with the source of a second TTSP-digraph is a TTSP-digraph.

The underlying undirected graph of a TTSP-digraph is called a *series-parallel graph*.

A TTSP-digraph G is naturally associated with a binary tree T, which is called the *decomposition tree* of G. The nodes of T are of three types, *Q-nodes*, *S-nodes*, and *P-nodes*, representing single edges, series compositions, and parallel compositions, respectively. The decomposition tree of the TTSP-digraph of Figure 2(a) is shown in Figure 2(b). It is well known that the decomposition tree of G has $O(n)$ nodes and can be constructed in $O(n)$ time [12].

Without loss of generality we assume that a child node of the same type of its parent is preferentially chosen as right child. Also, since G is a simple graph, any maximal set of connected P-nodes may have at most one Q-node child. Tree T can be rearranged in such a way that the Q-node is the left child of the top-most P-node, which we call a *final P-node*.

In the remaining part of this section we show that any TTSP-digraph admits a homothetic triangle contact representation, which can be computed in linear time.

Theorem 1 *Let G be a TTSP-digraph with n vertices. There exists an $O(n)$-time algorithm that computes a strict homothetic triangle contact representation of G.*

Proof sketch. Assume that G consists of more than one edge, otherwise the statement trivially holds.

We describe an algorithm that visits T from bottom to top and constructs a homothetic triangle contact representation Γ of G incrementally. Namely, let $\mu_1, \mu_2, \dots, \mu_k$ be the internal nodes (S- and P-nodes) of T ordered according to a post-order visit of T. Also, let $G_i \subseteq G$ denote the TTSP-digraph whose decomposition tree is the subtree rooted at μ_i ($1 \leq i \leq k$), and denote by s_i, t_i the source pole and the sink pole of G_i, respectively.

The drawing algorithm performs k steps; at step i ($1 \leq i \leq k$) a drawing Γ_i of G_i is computed. Notice that, since μ_k is the root of T, then $G_k = G$ and $\Gamma_k = \Gamma$. In Γ_i each vertex v is represented as a right triangle $\tau(v)$ with top corner a_v, bottom leftmost corner b_v, and bottom rightmost corner c_v. Also, $\tau(v)$ is such that the length of $\overline{b_v c_v}$ is equal to the length of $\overline{a_v b_v}$ (i.e., the angle $\widehat{b_v c_v a_v}$ is of 45 degrees). We call *size of* $\tau(v)$ the length of $\overline{b_v c_v}$.

Drawing Γ_i will have the following properties:

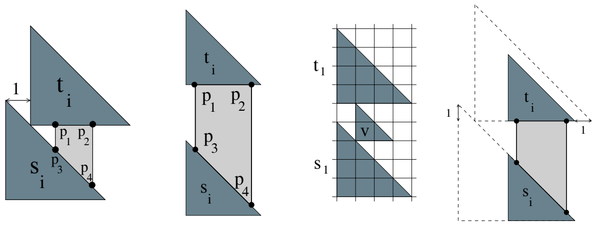
P1: Γ_i is a homothetic triangle contact representation of G_i.

P2: In Γ_i the triangles $\tau(s_i)$ and $\tau(t_i)$ have the same size; all triangles distinct from $\tau(s_i)$ and $\tau(t_i)$ are contained in a polygon with points p_1, p_2, p_3, p_4 where: $\overline{p_1 p_2}$ is properly contained in $\overline{b_{t_i} c_{t_i}}$; $\overline{p_3 p_4}$ is properly contained in $\overline{a_{s_i} c_{s_i}}$; p_1, p_3 (p_2, p_4) have the same x-coordinate. We call such a polygon the *inner polygon of* Γ_i.

P3: If μ_i is a final P-node, then drawing Γ_i has a shape like the one depicted in Figure 3(a), which we call an *α-shape*. Namely, b_{t_i} is the contact point of $\tau(s_i)$ and $\tau(t_i)$ and coincides with the point of $\overline{a_{s_i} c_{s_i}}$ that has horizontal distance 1 from $\overline{a_{s_i} b_{s_i}}$. Otherwise ($\mu_i$ is either an S-node or a non-final P-node), drawing Γ_i has a shape like the one depicted in Figure 3(b), which we call a *β-shape*. Namely, b_{s_i} and b_{t_i} have the same x-coordinate.

Since G is a simple graph, the base-case of the algorithm is an S-node μ_1 with two Q-nodes as children. Let (s_1, v) and (v, t_1) be the two edges that form G_1. We obtain a β-shape drawing Γ_1 for μ_1 satisfying Properties P1–P3 as depicted in Figure 3(c).

Assume by induction that Γ_j satisfies Properties P1–P3, for any $j < i$ and $i \geq 2$. We show how to construct Γ_i that also satisfies Properties P1–P3. We distinguish between three cases (we sketch the description for space reasons):



(a) (b) (c) (d)

Figure 3: (a) An α-shape. (b) A β-shape. (c) The base-case of the drawing algorithm. (d) A step of the algorithm.

- μ_i **is a final** P**-node:** The children of μ_i are a Q-node ν and a node μ_j ($j < i$) that is either an S-node or a non-final P-node. Hence, by the inductive hypothesis, Γ_j has a β-shape. We obtain Γ_i, which has an α-shape, by modifying Γ_j as depicted in Figure 3(d).

- μ_i **is a non-final** P**-node:** Denote by μ_j and μ_h ($j, h < i$) the children of μ_i. By definition of non-final P-node each of μ_j and μ_h is either an S-node or a non-final P-node, and therefore, by the inductive hypothesis, Γ_j and Γ_h have β-shapes. A β-shape drawing Γ_i can be obtained by placing Γ_j and Γ_h side by side, and by scaling the drawing of their inner polygons and of their poles.

- μ_i **is an** S**-node:** Let μ_j and μ_h ($j, h < i$) be the children of μ_i, where G_h is placed on top of G_j in the series composition. We distinguish between two main different cases: Γ_h has a β-shape or Γ_h has an α-shape. Assume first that Γ_h has a β-shape. If the size of $\tau(s_h)$ is smaller (resp., bigger) than the size of $\tau(t_j)$, then we place Γ_h on top of Γ_j identifying a_{s_h} with a_{t_j} (b_{s_h} with b_{t_j}, respectively) and then suitably scale $\tau(t_h)$ and $\tau(s_j)$ in order to restore Property `P2`. Conversely, assume that Γ_h has an α-shape. In this case, we first modify Γ_h in order to allow $\tau(t_h)$ to be scaled-up in the west direction and then apply the same strategy as above.

Concerning the time complexity, the post-order traversal of T takes $O(n)$ time and the operations required for each series- and parallel-composition can be computed in constant time. □

3 Contact Representations of Partial Planar 3-trees

A *planar* 3*-tree* is a graph obtained from a complete graph on 3 vertices by repeating the following operation:

- Choose a bounded triangular face and add a new vertex to this face connecting it to all 3 vertices of the chosen face.

The class of graphs does not change if the chosen face is also allowed to be the outer face, however, for our purpose the given definition is much more handy. Planar 3-trees are also known as *stacked triangulations*.

The construction of a triangle contact representation of a planar 3-tree can be obtained along the construction sequence of the graph. The part of the plane left uncovered by the 3 triangles of the initial 3 vertex graph consists of an unbounded region and a bounded region. The bounded region is a triangle which is homothetic to the shape of a point reflection of the triangle used to represent the vertices, let this be called a *triangle of co-shape*. Actually, the following strong property holds:

- Let C be a non-separating 3-cycle of a graph G. If G has a triangle contact representation, then the three triangles representing the vertices of C enclose a co-shaped empty triangle. The bounding edges of this empty triangle belong to the triangles representing the 3 vertices of C.

A triangle representing a vertex can be fitted inside a triangle of co-shape such that the corners touch the 3 bounding edges of the enclosing triangle. This is exactly the operation needed for the inductive construction of a strict triangle contact representation of a planar 3-tree along the construction sequence.

Theorem 2 *Every planar* 3*-tree with* n *vertices has a strict triangle contact representation, which is computable in* $O(n)$ *time.*

A graph G is a *partial planar* 3*-tree* if it is a subgraph of a planar 3-tree G^+, i.e., the graph G can be obtained by removing edges and vertices from G^+. We will show that a construction of a strict triangle contact representation of G^+ can be used to get a strict triangle contact representation of G.

The removal of vertices from G^+ is reflected by the removal of the corresponding triangles from the representation. Though the basic idea is again easy, the removal of edges is slightly more subtle. Recall that an edge is represented by a contact involving a corner of one triangle and a side of another. The idea for removing an edge is to slightly shrink the triangle contributing the corner and to simultaneously move it away so that the contacts of the other two edges are preserved. The plan is to go along a construction of a triangle contact representation of G^+ and to adapt size and placement of a triangle when it appears in the representation.

This idea can lead into problems if at some later stage the gap opened by an edge removal is too big compared to the size of a new triangle. Figure 4 should make this clear. Actually it is possible to place the triangle x such that it touches all of u, v, w, however, the contact between x and u would be of the wrong type making it impossible to place a common neighbor of u, v, x later.

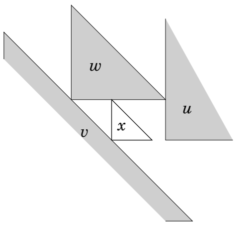


Figure 4: The gap opened to remove edge (u, v) turns out to be too big for x.

To avoid the problem we have to quantify the side length of the triangles. Without loss of generality we can assume that G^+ has n vertices and that the triangles used for the representation are equilateral. Let the side length of the co-shape enclosed by the first three vertices be A. The first (inner) triangle placed during the construction will have side length $A/2$ and the second has side length $A/4$. For further triangles we can give a bound: The kth triangle in the construction sequence of G^+ has side length $\geq A/2^k$. We choose $A = 2^n$, this makes the side length of all triangles involved powers of two and the side length of the smallest triangles ≥ 8 because there are only $n - 3$ inner triangles.

If a triangle which would have size B is to be shrunk, we reduce its side length by 1. Note that the side length of each of the 3 co-shapes formed by placing the resized triangle have side length $\geq B - 1$. Hence the size of a triangle fitting into one of these co-shapes is $\geq \frac{B-1}{2}$. Starting with the initial co-shape of side length A, we see that the kth triangle still has size $\geq (..((A - 1)\frac{1}{2} - 1)\frac{1}{2} \ldots - 1)\frac{1}{2} = \frac{1}{2^k}(A - 1 - 2 \ldots - 2^k) > \frac{A}{2^k} - 2$. With $A = 2^n$ we get no problems because all triangles have side length at least 6 and gaps are of size ≤ 1, hence, there is never the danger that a triangle could fit through a gap.

To actually compute a triangle contact representation for a partial planar 3-tree G, we need a corresponding host G^+. If G^+ is given, we can compute a construction sequence for G^+. Along this construction sequence a triangle contact representation is constructed which avoids contacts for all edges (u, v) of G^+ which are not in G but still the vertices u and v are in G. Given such a triangle contact representation, it remains to remove the triangles of vertices which do not belong to G. This yields a triangle contact representation of G.

Theorem 3 *Every partial planar 3-tree G has a strict triangle contact representation. If G is given together with a planar 3-tree G^+ which has G as a subgraph, a triangle contact representation of G can be computed in $O(n)$ time, where n is the number of vertices of G^+.*

Note that this result provides an alternative structural proof of Theorem 1, in view of the following proposition (the proof is omitted for space reasons).

Proposition 4 *Every series-parallel graph is a partial planar 3-tree.*

Proof sketch. The proof consists of showing by induction this statement: *Every series-parallel graph G with poles x, y is a subgraph of a plane triangulation G' with x, y on the boundary of the outerface, which can be reduced to the outside triangle by consecutive deletion of simplicial vertices of degree 3.* □

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