

Pointed Drawings of Planar Graphs

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Abstract

We study the problem how to draw a planar graph such that every vertex is incident to an angle greater than π . In general a straight-line embedding cannot guarantee this property. We present algorithms which construct such drawings with either tangent-continuous biarcs or quadratic Bézier curves (parabolic arcs), even if the positions of the vertices are predefined by a given plane straight-line embedding of the graph. Moreover, the graph can be embedded with circular arcs if the vertices can be placed arbitrarily. The topic is related to non-crossing drawings of multigraphs and vertex labeling.

1 Introduction

According to Fáry's theorem [3], every (simple) planar graph can be realized as plane straight-line embedding in the Euclidean plane. There is a vast literature dealing with the question of efficiently finding embeddings that fulfill certain (optimality) criteria. De Fraysseix, Pach and Pollack [4] and Schnyder [10] proved that every planar graph with n vertices can be drawn on a grid of size $(n-1) \times (n-1)$. The famous Koebe-Andreiev-Thurston circle packing theorem [1, 6, 11] states that every planar graph can be embedded in a way such that its vertices correspond to interior disjoint disks, which touch if and only if the corresponding vertices are connected with an edge, see also [7, 12].

Straight line embeddings forbid multiple edges between two vertices and loops. Thus, to draw multigraphs and graphs with loops, one has to allow more general edge shapes. One common way is to represent edges by polygonal lines, but another possibility is to choose smooth curves (for example the simplest type of curved curves, circular arcs).

A natural question is whether every planar multigraph can be drawn with circular arcs. Drawing parallel edges as circular arcs is no problem: an edge in a non-crossing straight-line drawing can be perturbed to any number of close-by circular arcs. Loops, however, require more space: The only circular arc between a ver-

tex and itself is a full circle through this vertex; thus, an angle of π incident to this vertex must be free of other emanating arcs. (This angle is then sufficient for any number of parallel loops.) This naturally leads to the question of pointed drawings of simple graphs without loops. In a *pointed drawing* of a graph, the incident edges of each vertex emanate within an open half plane.

Another potential application comes from drawing vertex labels. If the edges incident to a vertex point in all directions, it is hard to place a label close to its vertex. Thus it is good to have some angular space without emanating edges.

Haas et al. [5] showed that a planar graph has a plane pointed drawing with straight lines if and only if it is minimally rigid or a subgraph of a minimally rigid graph. A simple example of a graph that has no plane pointed embedding is the complete graph with four vertices.

We show that, for any given plane straight-line embedding of a planar graph, there exists a pointed plane embedding of the graph where the vertices stay where they are, and the edges are simple smooth curves of certain types: either quadratic Bézier curves, i. e., arcs of a parabola, or biarcs, i. e., curves consisting of two circular arcs that join in a tangent-continuous way. To the contrary, we observe that this is not possible when using only circular arcs as edges.

Further we prove that every planar graph has a plane pointed drawing with circular arcs as edges (for a non-fixed embedding of the vertices). Moreover, with tangent-continuous biarcs as edges, there exists a nice plane embedding such that for every point its incident edges all emanate in the very same direction (all sharing a common tangent). For quadratic Bézier curves as edges, the angle of edges emanating from a point can be made arbitrary small.

2 Definitions and Notation

Throughout this paper, let $G = (V, E)$ be a simple planar graph without loops, with finite vertex set V and finite set of edges E . In this paper we consider several types of plane embeddings $\mathcal{F}(G)$, all with some type of differentiable curves as edges.

For an embedding $\mathcal{F}(G)$ we denote the embedding of vertex $v \in V$ by $\mathcal{F}(v)$, and the embedding of an edge $e \in E$ by $\mathcal{F}(e)$.

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An embedding gives us a cyclic order of arcs leaving a vertex. The angle between two consecutive arcs is defined as the angle between the corresponding tangent rays.

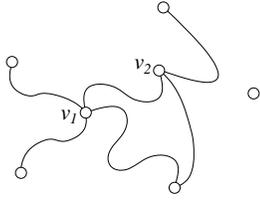


Figure 1: Embedding with a non-pointed vertex v_1 and a pointed vertex v_2 .

Definition 1 (Pointedness) A vertex in the embedding $\mathcal{F}(G)$ is called pointed if it is incident to an angle greater than π . If all vertices of a drawing are pointed we call the drawing pointed.

Definition 2 (ε -Pointedness) Let $\varepsilon > 0$. A vertex in the embedding $\mathcal{F}(G)$ is called ε -pointed if it is incident to an angle greater than $2\pi - \varepsilon$. If all vertices of a drawing are ε -pointed we call the drawing ε -pointed.

For the special case of straight-line embeddings, this definition is identical to the classic definition of pointedness, see [8, 9].

3 Starting from a Plane Straight-Line Embedding

Theorem 1 Let a planar graph $G = (V, E)$ and a straight-line embedding $\mathcal{F}_s(G)$ be given. There exists a pointed embedding $\mathcal{F}_q(G)$ with quadratic Bézier curves as edges such that $\mathcal{F}_q(v) = \mathcal{F}_s(v)$ for all $v \in V$.

Moreover, for every $v \in V$ the cyclic order of the edges incident to v in $\mathcal{F}_s(G)$ is the same as in $\mathcal{F}_q(G)$.

Proof. Assume that in $\mathcal{F}_s(G)$, no two vertices have identical x -coordinates or y -coordinates. We build up $\mathcal{F}_q(G)$ from bottom to top. To this end we order the vertices by y -coordinates s.t. $y(v_i) < y(v_j), \forall i < j$. We start by embedding all edges that are incident to v_1 as straight lines. As $\mathcal{F}_s(v_1)$ lies on the convex hull of $\mathcal{F}_s(G)$, $\mathcal{F}_q(v_1)$ is pointed. Now when processing vertex $v_i \in V, i \geq 2$, all edges that connect v_i to a vertex below v_i in $\mathcal{F}_s(G)$ are already embedded. All these edges emanate from $\mathcal{F}_s(v_i)$ strictly below the horizontal line h through $\mathcal{F}_s(v_i)$. The remaining edges, which point upwards, emanate either to the left or to the right. We process the edges on each side separately.

Suppose there are k edges that point upward and to the right. We process these edges from bottom to top, i.e., in counter-clockwise order e_1, \dots, e_k . We want to embed $e_1 = \overline{v_i v_j}$ as a quadratic Bézier curve emanating

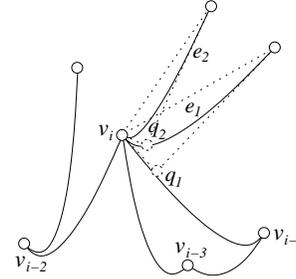


Figure 2: Construction of a plane pointed embedding where the edges are quadratic Bézier curves.

from $\mathcal{F}_s(v_i)$ below h . To this end we choose a narrow triangle $\mathcal{F}_s(v_i)\mathcal{F}_s(v_j)q_1$, as shown in Figure 2.

The triangle must contain no vertices, and the point q_1 must lie below h , but above all already embedded edges $\mathcal{F}_q(\overline{v_l v_i})$ at $\mathcal{F}_q(v_i)$. We embed e_1 as the Bézier curve from $\mathcal{F}_s(v_i)$ to $\mathcal{F}_s(v_j)$ that uses this triangle as a control polygon. The tangent directions at the endpoint are the edges of the triangle, and thus, $\mathcal{F}_q(e_i)$ emanates from $\mathcal{F}_q(v_i)$ below h . For embedding an edge $e_i, 2 \leq i \leq k$, we proceed in a similar way. We place the control point q_i below h and on $\mathcal{F}_q(e_{i-1})$, thus ensuring that $\mathcal{F}_q(e_{i-1})$ and $\mathcal{F}_q(e_i)$ do not cross, and that $\mathcal{F}_q(e_i)$ still emanates from $\mathcal{F}_q(v_i)$ below h .

The edges pointing left are treated in an analogous manner. □

Theorem 2 For every graph $G = (V, E)$ with a given plane straight-line embedding $\mathcal{F}_s(G)$, there exists a pointed embedding $\mathcal{F}_b(G)$ with tangent-continuous biarcs as edges such that $\mathcal{F}_b(v) = \mathcal{F}_s(v)$ for all $v \in V$.

Moreover, for every $v \in V$ the cyclic order of the edges incident to v in $\mathcal{F}_s(G)$ is the same as in $\mathcal{F}_q(G)$.

Proof. Our construction will mimic the proof of Theorem 1. Again, we build up the embedding \mathcal{F}_b from bottom to top, considering the not yet embedded incident edges for each vertex $v_i \in V, i \geq 2$ in the same order. Like before, consider a narrow triangle

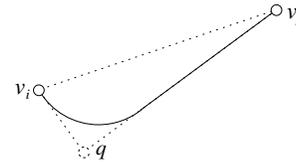


Figure 3: Embedding of an edge as a tangent-continuous biarc in a triangle.

$\mathcal{F}_s(v_i)\mathcal{F}_s(v_j)q$ for each edge $\overline{v_i v_j}$ that we want to embed. Instead of a quadratic Bézier curve, we take a circular arc that is tangent to $\mathcal{F}_s(v_i)q$ at v_i and to $q\mathcal{F}_s(v_j)$, and complete the edge with a straight line along $q\mathcal{F}_s(v_j)$, see Figure 3. (When speaking of biarcs or

more generally of circular arcs, we always allow straight line segments as special cases.) This construction assumes that $\mathcal{F}_s(v_i)q$ is shorter than $q\mathcal{F}_s(v_j)$. Otherwise, a symmetric construction works.

With the same arguments as in the proof of Theorem 1, this leads to a plane, pointed embedding $\mathcal{F}_b(G)$. \square

Theorem 3 *There are plane graphs $G = (V, E)$ and straight-line embeddings $\mathcal{F}_s(G)$, for which there are no pointed embeddings $\mathcal{F}_c(G)$ with circular arcs as edges such that $\mathcal{F}_c(v) = \mathcal{F}_s(v)$ for all $v \in V$.*

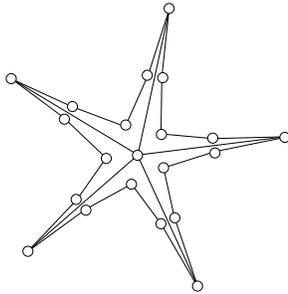


Figure 4: Starfish example of a plane embedding that cannot be drawn pointed with circular arcs as edges.

Proof. Consider the plane straight-line graph shown in Figure 4. The five points close to the center restrict the circular arcs that are used for the edges incident to the center. Since the arms of the star are thin enough, there is no way to make the central vertex pointed. To generalize this example to a larger number of points, simply give the starfish more arms. \square

4 Starting from an Abstract Planar Graph

In the last section we restricted ourselves to a predefined placement of the points, determined by a given plane straight-line embedding. If the location of the vertices can be chosen arbitrarily, we get the following easy consequence of Theorem 1.

Theorem 4 *For any $\varepsilon > 0$ and any planar graph G , there exists an ε -pointed embedding $\mathcal{F}_q(G)$ with quadratic Bézier curves as edges.*

Proof. Consider an arbitrary straight-line embedding $\mathcal{F}_s(G)$. In the proof of Theorem 1 we showed a construction for a pointed embedding $\mathcal{F}'_q(G)$, in which all vertices point either to the bottom or to the top. By squeezing the embedding $\mathcal{F}_s(G)$ in direction of the x -axis, the control triangles, which determine the tangent directions, approach the vertical direction arbitrarily closely. Thus, we obtain an embedding $\mathcal{F}_q(G)$ where for every vertex $v \in V$, all edges emanate within an angle of ε . \square

By the same arguments, it is possible to apply the construction in the proof of Theorem 2 to obtain an ε -pointed embedding $\mathcal{F}_b(G)$ with biarcs. The disadvantage of this approach is that the obtained embeddings tend to have a bad aspect ratio. Therefore, in the following we present a stronger and nicer result for biarcs.

Theorem 5 *Every planar graph $G = (V, E)$ has a pointed embedding $\mathcal{F}_b(G)$ with tangent-continuous biarcs as edges such that $\mathcal{F}_b(G)$ is ε -pointed for any $\varepsilon > 0$. That is, for all vertices $v \in V$ all edges incident to v share a common tangent at $\mathcal{F}_b(v)$ in $\mathcal{F}_b(G)$. The directions of these tangents can be specified independently for each vertex.*

Proof. According to the Koebe-Andreev-Thurston circle packing theorem [1, 6, 11], there exists a straight-line embedding $\mathcal{F}_s(G)$ such that the vertices of $v \in V$ are embedded as the center points of disjoint disks. Moreover, two such disks touch if and only if the corresponding vertices are connected with an edge in G .

We start by choosing such an embedding for the vertices. We place every vertex $v \in V$ on an arbitrary point of the boundary of the disk corresponding to v in $\mathcal{F}_s(G)$, avoiding touching points of the disk.

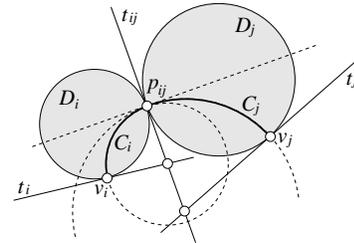


Figure 5: Construction of a tangent-continuous biarc from two touching disks D_i, D_j .

For any edge $e = \overline{v_i v_j} \in E$, let p_{ij} be the touching point of their corresponding disks D_i, D_j , see Figure 5. We draw a circular arc from v_i to p_{ij} inside D_i , as part of the circle through v_i and p_{ij} that is orthogonal to D_i , and we continue inside D_j to v_j , on an arc orthogonal to D_j . The arcs have a common tangent at the point p_{ij} where they meet, orthogonal to the common tangent of D_i and D_j at this point. Since the arcs inside each disk D_i share a common tangent at v_i , they don't cross, and we have a plane embedding. \square

The above proof leaves some freedom to place the vertices on the boundaries of the related disks. If in the embedding $\mathcal{F}_s(G)$ no two disk centers have the same x -coordinate, we can place each vertex on the bottommost point of the boundary of its disk. We then obtain a drawing where all vertices are pointed downwards, see Figure 6 for an example. By this, both arcs of the edges

bend in the same direction. (There are no S-shaped biarcs.) Another possibility is to place each vertex $v_i \in$

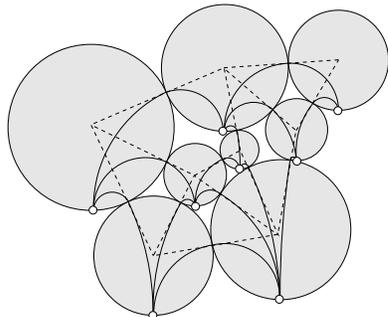


Figure 6: Pointed embedding with biarcs as edges, constructed from a circle packing.

V farthest away from any touching point of its disk D_i . In this way we can guarantee the radius of any circular arc inside D_i to be at least $R_i \cdot \tan \frac{\pi}{2k_i}$, where R_i is the radius of D_i , and $k_i \geq 2$ is the degree of v_i . See the full paper for more details. Finally, let us remark that the freedom in placing the vertices allows us to choose a direction of pointedness for each vertex separately.

Theorem 6 *Every planar graph has a pointed embedding with circular arcs as edges.*

The basic idea of this embedding is based on the canonical representation for plane graphs [4], which is a labeling of the vertices of G meeting several conditions. Using this labeling, we build up a flat embedding of G where all vertices are pointed to the bottom. For the proof refer to the full version of the paper.

A different way to find a pointed embedding utilizes the framework established in [5]. We can transform the abstract graph G into a so-called *combinatorial pointed pseudo-triangulation* by subdividing at most $n-3$ edges. With help of the techniques introduced in [5, Theorem 5.1] we obtain a pointed polygonal embedding of the modified graph. The drawing has one bend for every subdivided edge. The bends can be easily replaced by biarcs or quadratic Bézier curves.

Theorem 7 *Every planar graph with n vertices has a pointed embedding with either quadratic Bézier curves, biarcs, or polygonal chains consisting of two line segments, which uses at most $n-3$ non-straight edges. Moreover, for each inner vertex, one can arbitrarily choose a face in which it is pointed. For the case of Bézier curves and polygonal chains, it is also possible to prescribe the shape of each face up to affine transformations.*

The complete proof will be given in the full version of the paper. In general it is not possible to draw a pointed embedding with a larger number of straight lines. In this sense, Theorem 7 is optimal.

5 Conclusion

We have shown that every plane straight-line embedding can be redrawn pointed with identically embedded vertices, and either tangent-continuous biarcs or quadratic Bézier curves as edges. We can even ensure that all edges emanate from the vertices within an arbitrary small angle. These drawings are probably not satisfactory from an aesthetic point of view. Our embedding with biarcs, where all incident edges share a common tangent, is nicer and may be more useful for applications. Still, as the construction we use relies on circle packings (which to compute is considered a hard problem), it is an interesting question whether Theorem 5 can also be proven without using circle packings. Further, it remains open whether there exist aesthetically nice embeddings also for Bézier curves, or even for circular arcs.

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