Practical C^1 reparametrization of piecewise rational Bézier curves

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Abstract

Piecewise rational Bézier curves with G^1-continuity in the projective space provide a useful tool for shape blending applications with complex boundary conditions where visual continuity is required. This paper presents an efficient method to construct such curves given an initial sequence of rational segments G^1-continuous in the affine space. Combination of degree elevation and linear rational reparametrization is used to transform the curve in the projective space.

1 Introduction

Shape blending and skinning are common geometrical design operations, which are often performed by a linear interpolation of a pair of piecewise polynomial curves defining the source and target shapes [1], or by interpolating control polygon vertices. The problem of retaining visual continuity in shape blending and skinning methods which use piecewise rational curves was addressed by different authors. In [2], the approach of adjusting the junction points was used to retain G^1-continuity. Another approach is to use a projective space transformation (such as a linear rational reparametrization) to transform the segments required to be G^1-continuous in the projective space [4].

It is well-known that any linear rational reparametrization (also known as Mœbius transform) of a Bézier curve $t \to \hat{t},\ \hat{\mathbf{C}}(\hat{t}) = \mathbf{C}(t),\ t = b\hat{t}/(1+(b-1)\hat{t})$ is equivalent to multiplying the weights of the control points [4] by some coefficients forming a geometrical progression: $\hat{w}_i = b^i w_i, \quad i = 0,\dots,n$. Hence, by changing the weights of the control points of individual segments, the tangent vectors at the joint points can be aligned and the final composite curve may be easily parametrized as a C^1-continuous curve with parameter in $[0,1]$ for interpolation. A B-spline representation can be obtained if needed by passing to the B-spline basis and then removing the joint control points and their knot values (see for example [3]). However a well-defined *composite reparametrization* does not always exist. By using degree elevation the needed degrees of freedom can be added to make reparametrization possible in all situations.

We will denote the projective space by $\mathbb{P}$ and the associated affine space by E. Then $G^k(\mathbb{P})$ and $G^k(E)$ denote k-degree continuity with respect to the corresponding space. We will use lower-case letters to denote projective coordinates and upper-case letters for affine coordinates. A 2D affine space is assumed in this paper, and hence the corresponding $\mathbb{P} = \mathbb{P}^2$ appears as 3D, but the results can be directly extended to higher dimensions. *Reparametrization* is assumed to mean *linear rational reparametrization* unless explicitly specified.

We will first present an analytic solution to the composite normalization problem, and then address the alignment and degree elevation problems.

2 Composite reparametrization

Let a piecewise rational Bézier curve be defined by a sequence of m segments $\mathbf{C}_k(t),\ t \in [0,1],\ k = 1,\dots,m$,

$$\mathbf{C}_k(t) = \frac{\sum_{i=0}^{n_k} B_{n_k,i}(t) w_{k,i} \mathbf{P}_{k,i}}{\sum_{i=0}^{n_k} B_{n_k,i}(t) w_{k,i}}$$

where each segment k is of degree n_k and $B_{n_k,i}$ denotes the Bernstein basis function $B_{n_k,i}(t) = \binom{n_k}{i}(1-t)^{n_k-i}t^i$. The corresponding polynomial curve is defined in the projective space as

$$\mathbf{c}_k(t) = \sum_{i=0}^{n_k} B_{n_k,i}(t)\mathbf{p}_{k,i}, \quad \mathbf{p}_{k,i} = (w_{k,i}\mathbf{P}_{k,i} | w_{k,i}) \,.$$

We will consider a G^1 affine curve while the projective curve is at least G^0, in such a way that $\mathbf{p}_{k,n_k} = \mathbf{p}_{k+1,0},\ k = 1,\dots,m-1$. The segments are assumed to have no singularities.

2.1 Linked transforms

Suppose the composite curve is $G^1(\mathbb{P})$. Let $b_i,\ i = 1,\dots,m$ stand for some positive coefficients defining the reparametrization of its segments. We study the conditions for b_i to retain continuity in the projective space. A trivial example of a valid b_i sequence is the unit transform: $b_1 = b_2 = \dots = b_m = 1$.

Consider the first two segments of a composite curve and the corresponding control vertices adjacent to the

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joint point $\mathbf{p}_{1,n_1} = \mathbf{p}_{2,0}$. Let $\mathbf{p}_{2,-} = \mathbf{p}_{1,n_1-1}$, $\mathbf{p}_{2,\circ} = \mathbf{p}_{2,0}$, $\mathbf{p}_{2,+} = \mathbf{p}_{2,1}$. These points transform as [1]

$$\begin{aligned}\hat{\mathbf{p}}_{2,-} &= b_1^{n_1-1}\mathbf{p}_{2,-}\\ \hat{\mathbf{p}}_{2,\circ} &= b_1^{n_1}\mathbf{p}_{2,\circ}\\ \hat{\mathbf{p}}_{2,+} &= b_1^{n_1} b_2 \mathbf{p}_{2,+}\,.\end{aligned} \tag{1}$$

From $G^1(\mathbb{P})$-continuity, $\mathbf{p}_{2,-}, \mathbf{p}_{2,\circ}$ and $\mathbf{p}_{2,+}$ must be collinear, in such a way that $\exists \xi_1 \in (0,1)$ such that

$$\mathbf{p}_{2,\circ} = \xi_1 \mathbf{p}_{2,-} + (1-\xi_1)\mathbf{p}_{2,+} \tag{2}$$

and similarly $\exists \eta_1 \in (0,1)$ such that

$$\hat{\mathbf{p}}_{2,\circ} = \eta_1 \hat{\mathbf{p}}_{2,-} + (1-\eta_1)\hat{\mathbf{p}}_{2,+}\,. \tag{3}$$

After substitution of (1) into (3) and division by $b_1^{n_1}$, one gets

$$\mathbf{p}_{2,\circ} = \frac{\eta_1}{b_1}\mathbf{p}_{2,-} + (1-\eta_1) b_2 \mathbf{p}_{2,+}\,. \tag{4}$$

From (2) and (4), $\eta_1/b_1 = \xi_1$, $(1-\eta_1)b_2 = (1-\xi_1)$, and finally

$$b_2 = \frac{1-\xi_1}{1-\xi_1 b_1}\,. \tag{5}$$

In general, a recurrence relation is obtained:

$$b_{k+1} = \frac{1-\xi_k}{1-\xi_k b_k} \quad \text{where} \quad \xi_k = \frac{\|\mathbf{p}_{k,\circ}\mathbf{p}_{k,+}\|}{\|\mathbf{p}_{k,-}\mathbf{p}_{k,+}\|}\,, \tag{6}$$

in such a way that all the other coefficients can be found by fixing b_1. The sequence $\{b_k\}$ obtained by using (6) is called a *linked transform*. A linked transform is said to be *well-defined* if all b_k are positive. It is easy to prove that, for all well-defined linked transforms, $b_1 \in (0, b_{1max})$ where b_{1max} can be found by using the inverse of (6) applied to a sequence starting from $b_m = 0$.

Developing recurrence (6), b_{k+1} can be written as a finite continued fraction:

$$b_{k+1} = \cfrac{1-\xi_k}{1-\cfrac{\xi_k(1-\xi_{k-1})}{1-\cfrac{\xi_{k-1}(1-\xi_{k-2})}{\ddots\, 1-\cfrac{\xi_2(1-\xi_1)}{1-\xi_1 b_1}}}}\,. \tag{7}$$

2.2 Normalization

A rational Bézier curve of degree n is said to be in standard form if $w_0 = w_n = 1$. Similarly, a composite curve is called *normalized* if it is $G^1(\mathbb{P})$ and $w_{1,0} = w_{m,n_m} = 1$.

Theorem 1 *For a given $G^1(\mathbb{P})$-continuous composite curve, there exists a unique linked transform $\{b_k\}$ which transforms it to a normalized form.*

[1] The factor $b_1^{n_1}$ in $\hat{\mathbf{p}}_{2,+}$ is needed to retain G^0-continuity.

Proof. The first weight $w_{1,0}$ can be simply normalized by applying a uniform scaling $1/w_{1,0}$ to all vertices, in such a way that it can be assumed that $w_{1,0} = 1$. A linked transform $\{b_k\}$ normalizes the last weight if

$$w_{m,n_m} \prod_{k=1}^{m} b_k^{n_k} = 1\,. \tag{8}$$

It is easy to show that, by varying b_1 in the admissible interval $(0, b_{1max})$, the product $\prod_{k=1}^{m} b_k^{n_k}$ can take any positive value, and hence a solution of (8) always exists. This solution is also unique because each relation (6) defines a monotonous function $b_{k+1}(b_k)$. □

Consider now the case of all the segments having the same degree n. Then (8) can be simplified as

$$\prod_{k=1}^{m} b_k = w_{m,n}^{-1/n} \tag{9}$$

and an analytic solution can be found by using a technique based on the continued fraction theory. Let $b_k = P_k/Q_k$. From (7),

$$b_{k+1} = \frac{(1-\xi_k)Q_k}{Q_k - \xi_k P_k} = \frac{P_{k+1}}{Q_{k+1}}\,. \tag{10}$$

Suppose b_1 is known. We can define the following recurrence relations:

$$\begin{aligned}P_{k+1} &= (1-\xi_k)Q_k\\ Q_{k+1} &= Q_k - \xi_k P_k\\ P_1 &= b_1\,, \quad Q_1 = 1\,.\end{aligned} \tag{11}$$

After substituting (11) into (10), equation (9) becomes

$$\frac{b_1}{Q_1} \cdot \frac{(1-\xi_1)Q_1}{Q_2} \cdot \ldots \cdot \frac{(1-\xi_{m-1})Q_{m-1}}{Q_m} = w_{m,n}^{-1/n}$$

in such a way that, after cancelling the common terms, the following equation is obtained for b_1:

$$b_1 = \frac{Q_m\, w_{m,n}^{-1/n}}{\prod_{k=1}^{m-1}(1-\xi_k)}\,, \tag{12}$$

where Q_m is by construction a linear function of b_1 which in turn can be expressed by using recurrences:

$$\begin{aligned}Q_k &= A_k b_1 + B_k\\ A_{k+1} &= A_k - \xi_k(1-\xi_{k-1})A_{k-1}\\ B_{k+1} &= B_k - \xi_k(1-\xi_{k-1})B_{k-1}\\ A_1 &= 0,\ A_2 = -\xi_1,\ B_1 = B_2 = 1\,.\end{aligned} \tag{13}$$

After substituting (13) into (12) we finally obtain

$$b_1 = \frac{B_m}{w_{m,n}^{1/n}\prod_{k=1}^{m-1}(1-\xi_k) - A_m}\,, \tag{14}$$

where A_m and B_m only depend on ξ_k. The resulting sequence b_k found from the recurrence relation will normalize the composite curve.

Theorem 2 *The sequence obtained by solving* (12) *is well-defined for any set of* $\xi_k \in (0,1)$.

The proof is very technical and given in the full version of the paper.

Analytic normalization requires all the segments to have the same degree, otherwise some terms in (8) will have different powers and cannot be simplified, in such a way that the resulting equation will be a polynomial equation on b_1. Theorem 1 guarantees that it has exactly one solution in the interval $(0, b_1^{max})$, which can be found using numerical methods. In practical applications, composite curves are often constructed from segments of the same degree, or which have their degree between 1 and some fixed N, in which case all the segment degrees can be elevated to degree N and (14) can still be used.

2.3 Alignment problem

Consider a composite $G^1(E)$ curve which is not $G^1(\mathbb{P})$. Such a curve is called *non-aligned*, as opposed to *aligned* $G^1(\mathbb{P})$ curves.

In this section, we will show that, by using appropriate linear rational reparametrization of the segments, it is possible to obtain $G^1(\mathbb{P})$-continuity under some conditions. For this it is sufficient to "align" the vectors defined by the joint point and the adjacent control vertices (figure 1), since the tangents to the Bézier segments are proportional to these vectors. Since reparametrization results in moving the control vertices along the dashed lines, it is possible to make them collinear. Geometrically, the reparametrization can be defined by fixing for example $\hat{\mathbf{p}}_-$ and finding $\hat{\mathbf{p}}_+$ as the intersection of $\hat{\mathbf{p}}_-\hat{\mathbf{p}}_0$ and $\mathbf{0}\hat{\mathbf{p}}_+$. All the points are coplanar since the curve is $G^1(E)$. Of course, not all positions of $\hat{\mathbf{p}}_-$ are possible, since for some of them the intersection does not exist, or is in the half-space of negative w.

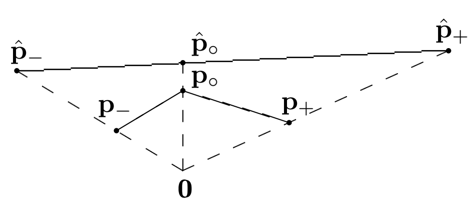


Figure 1: Alignment of control vertices

Alignment is always possible for two-segment curves, since there is only one joint point, in such a way that line $\hat{\mathbf{p}}_-\hat{\mathbf{p}}_+$ can be made horizontal. Since reparametrization affects all the control vertices of a segment, alignment of any two segments of a composite curve fixes the reparametrization coefficients for all the other segments. It follows that in some cases it is impossible to apply this method to a multi-segment curve [4].

To study the conditions under which alignment is analytically possible, we introduce the *central inclination* function $\theta(t)$ of a rational parametric curve $\mathbf{c}(t)$ with affine projection $\mathbf{C}(t)$ as

$$\theta(t) = \frac{w'(t)}{\|\mathbf{C}'(t)\| w(t)} = \frac{w'(t)}{\|\mathbf{c}'(t) - (\mathbf{C}(t)|1)w'(t)\|}\,, \quad (15)$$

where the norms are euclidean in the corresponding spaces. In addition, $\phi(t) = \arctan(\theta(t))$ is defined as the *central inclination angle* of $\mathbf{c}(t)$.

To explain the geometrical meaning let us fix some $t = t^*$ and translate the curve using affine translation of vector $\mathbf{V} = -\mathbf{C}(t^*)$. The translated affine curve defined as $\hat{\mathbf{c}}(t) = \mathbf{c} + (\mathbf{V}w(t), w(t))$ will then pass by the origin, $\hat{\mathbf{C}}(t^*) = \mathbf{0}$. Since translation does not change the affine derivative nor the weight,

$$\hat{\theta}(t^*) = \theta(t^*) = \frac{w'(t)}{\|\hat{\mathbf{C}}'(t)\| w(t)} = \frac{w'(t^*)}{\|\hat{\mathbf{c}}'(t^*)\|} = \frac{w'(t^*)}{\|\hat{\mathbf{c}}'(t^*)\|}\,,$$

and $\theta(t^*)$ is the tangent of the angle between the tangent vector $\hat{\mathbf{c}}'(t^*)$ and the horizontal hyperplane $w = w(t^*)$.

It is easy to check that central inclination is invariant not only to affine translations, but also to affine rotations and uniform scaling in $\mathbb{P}$. It is also invariant to linear reparametrization, meaning that if $\hat{t} = \lambda t$ then $\hat{\theta}(\hat{t}) = \theta(t)$. For composite rational curves, this also means that $G^1(\mathbb{P})$-continuity requires C^0-continuity of $\theta(t)$.

Using the well-known formula for the tangent vectors to Bézier curves,

$$\mathbf{c}'(0) = n(\mathbf{p_1} - \mathbf{p_0})\,, \quad \mathbf{c}'(1) = n(\mathbf{p_n} - \mathbf{p_{n-1}})\,, \quad (16)$$

it is easy to express the endpoint central inclinations. Substituting (16) into (15) and using $\|(\mathbf{P}|1) - (\mathbf{Q}|1)\| = \|\mathbf{P} - \mathbf{Q}\|$, one gets

$$\theta^{(0)} = \theta(0) = \frac{w_1 - w_0}{w_1\|\mathbf{P}_1 - \mathbf{P}_0\|}$$
$$\theta^{(1)} = \theta(1) = \frac{w_n - w_{n-1}}{w_{n-1}\|\mathbf{P}_{n-1} - \mathbf{P}_n\|}\,.$$

Let $L^{(0)} = 1/\|\mathbf{P}_1 - \mathbf{P_0}\|$, $L^{(1)} = 1/\|\mathbf{P}_{n-1} - \mathbf{P}_n\|$. Then

$$\theta^{(0)} = L^{(0)}\frac{w_1 - w_0}{w_1}\,, \quad \theta^{(1)} = L^{(1)}\frac{w_n - w_{n-1}}{w_{n-1}}\,.$$

The inclinations after a reparametrization with coefficient b now write as follows:

$$\theta^{(0)}(b) = \frac{w_0 L^{(0)}}{w_1}\left(\frac{w_1}{w_0} - \frac{1}{b}\right)$$
$$\theta^{(1)}(b) = \frac{w_n L^{(1)}}{w_{n-1}}\left(b - \frac{w_{n-1}}{w_n}\right)\,.$$

The graph of $(\theta^{(0)}(b), \theta^{(1)}(b))$ for $b > 0$ is easily found to be a branch of hyperbola with asymptotes $L^{(0)}$, $-L^{(1)}$.

This leads to the following algorithm to test if alignment of the composite curve is possible: Start with $\theta_1 = -L_1^{(1)}$. At step k, $k = 2, \ldots, m-1$, find θ_k as

$$\theta_k = L_k^{(1)} \left(\frac{L_k^{(0)} w_{k,0} w_{k,n_k}}{(L_k^{(0)} - \theta_{k-1}) w_{k,1} w_{k,n_k-1}} - 1 \right) .$$

If $\theta_k < L_{k+1}^0$ then proceed, otherwise the curve cannot be aligned.

3 Combining degree elevation with reparametrization

It can be shown that it is impossible to align three quadratic segments representing 90° circular arcs whose radii are $1/2, 1$ and $1/2$ by using reparametrization, while this becomes possible by elevating them to cubics. The solution is to combine degree elevation with reparametrization using different parameters for each degree.

Consider the reparametrization of an n-degree curve with parameter c, followed by a canonical degree elevation and then by another reparametrization with parameter b. The new vertices q_i, $i = 0, \ldots, n+1$, are

$$\mathbf{q}_0 = \mathbf{p}_0 , \quad \mathbf{q}_{n+1} = b^{n+1} c^n \mathbf{p}_n$$
$$\mathbf{q}_i = b^i \frac{i c^{i-1} \mathbf{p}_{i-1} + (n+1-i) c^i \mathbf{p}_i}{n+1}, \; i = 1, \ldots, n .$$

Without loss of generality the analysis can be performed with the original curve in standard form. Let $l^0 = L^{(0)}/nw_1$, $l^1 = L^{(1)}/nw_{n-1}$. The resulting endpoint central inclinations are:

$$\theta^{(0)}(c,b) = L^{(0)} + l^{(0)} \left(\frac{1}{c} - \frac{n+1}{bc} \right) \tag{17}$$
$$\theta^{(1)}(c,b) = -L^{(1)} - l^{(1)} \left(c - (n+1)bc \right) . \tag{18}$$

The graph of (17), (18) is a 2D area defined by a branch of hyperbola and two lines.

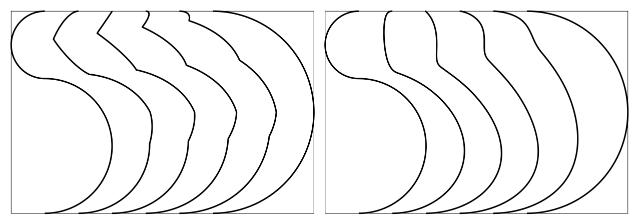
One particular useful case of alignment is zero-inclination alignment, with $\theta^{(0)}(c,b) = \theta^{(1)}(c,b) = 0$. System (17),(18) has two solutions:

$$c_{1,2}(x,y) = \frac{1 - (n+1)^2 + n^2 w_1 w_{n-1} \mp \sqrt{D}}{-2nw_1} \tag{19}$$
$$b_{1,2}(x,y) = \frac{1 + (n+1)^2 - n^2 w_1 w_{n-1} \mp \sqrt{D}}{2(n+1)} , \tag{20}$$

where $D = (1 + (n+1)^2 - n^2 w_1 w_{n-1})^2 - 4(n+1)^2$. From the analysis of (19), (20), it follows that the solutions are only positive for $w_1 w_{n-1} \leq 1$. The choice of a particular solution can be made depending on some optimization criterion for the composite curve (for example as to minimize the w variation). In the case of conic segments, $w_1 = w_{n-1} \leq 1$ is satisfied for elliptical arcs and parabolic segments. Therefore we can easily align a composite curve composed of such segments together with polynomial cubics and the final piecewise cubic curve can be normalized using the previous section results. This is an important practical case for CAD applications.

4 Examples

Figure 2 presents a test of the alignment method proposed. The left sequence of curves is a linear blending of the left curve composed of four circular arcs and the right curve composed of two circular arcs, where the segments are taken in standard form and hence are not aligned. The global parametrization of the curves is C^1 in the affine space. Here the kinks due to the discontinuous weight function are visible. On the other hand, the right sequence results from the blending of $C^1(\mathbb{P})$-continuous normalized curves after performing reparametrization combined with degree elevation.



(a) (b)

Figure 2: Interpolation in the projective space (a) non-aligned segments; (b) after segment alignment;

5 Conclusion

We have presented a practical method to construct G^1-continuous composite rational curves which can be used in blending or skinning applications with visual continuity requirement.

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