Fast computation of smallest enclosing circle with center on a query line segment

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Abstract

Here we propose an efficient algorithm for computing the smallest enclosing circle whose center is constrained to lie on a query line segment. Our algorithm preprocesses a given set of n points $P = \{p_1, p_2, \ldots, p_n\}$ such that for any query line or line segment L, it efficiently locates a point c' on L that minimizes the maximum distance among the points in P from c'. Roy et al. [11] has proposed an algorithm for this problem that reports the location of the center of the smallest enclosing circle C' on a query line segment in $O(\log^2 n)$ time. Our algorithm improves the query time compared to [11]. The reporting time for this problem is $O(\log n)$ and both the preprocessing time and space complexities are $O(n^2)$.

1 Introduction

The problem of enclosing a set of points with a minimum radius circle was originally posed in 1857 by Sylvester [13]. Here the objective is to report the center of the minimum radius circle which can enclose all the points in P. Elzinga and Hearn [4] first proposed an $O(n^2)$ time algorithm. Later, Shamos and Hoey [12], Preparata [9] and Lee [7] independently proposed $O(n \log n)$ time algorithms to solve this problem. Finally Megiddo [8] proposed an optimal $O(n)$ time algorithm using prune-and-search technique.

Megiddo [8] studied the constrained case of this problem where the center of the smallest enclosing circle of P lies on a given straight line. He gave an $O(n)$ time solution. Hurtado et. al. [6] and Bose et al. [2] considered the problem where the center of the smallest enclosing circle of P is constrained to lie inside a given simple polygon of size m. Their proposed algorithm runs in $O((n+m)\log(n+m)+k)$ time, where k is the number of intersections of the boundary of the polygon with the furthest point Voronoi diagram of P. In the worst case, k may be $O(n^2)$. This result is later improved to $O((n+m)\log m + m \log n)$ [3]. In particular, if the polygon is a convex one, then the problem can be solved in $O((n+m)\log(n+m))$ time [2]. In a further generalization of this problem, r (≥ 1) simple polygons with a total of m vertices are given; one of them can contain the center of the smallest enclosing circle of the point set P [3]. The time complexity of this version is $O((n+m)\log n + (n\sqrt{r}+m)\log m + m\sqrt{r} + r^{\frac{3}{2}} \log r)$. The query version of this problem is studied by Roy et al. [11]. Their proposed algorithm reports the center and the radius of the smallest enclosing circle whose center is constrained to lie on a query line segment in $O(\log^2 n)$ time. The preprocessing time and space complexity of their algorithm are $O(n \log n)$ and $O(n)$ respectively.

In this work, we will consider the problem posed in [11]. We will reduce the query time complexity to $O(\log n)$ by using the technique of geometric duality. But the preprocessing time and space complexity required are both $O(n^2)$.

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2 Constrained 1-center problem

Given a point set $P = \{p_1, p_2, \ldots, p_n\}$ and a query line L, our objective is to enclose P with a minimum radius circle C' whose center c' is constrained to lie on L. Note that, the smallest circle enclosing the vertices of the convex hull of a point set P will also enclose all the points in P. So instead of considering the whole point set, here we consider P as a convex polygon with vertices $\{p_1, p_2, \ldots, p_n\}$ in clockwise order.

2.1 Basic Results

The furthest point Voronoi diagram $\mathcal{V}(P)$ of point set P partitions the plane into n unbounded convex regions, namely $R(p_0), R(p_1), \ldots, R(p_{n-1})$, such that for any point $p \in R(p_j)$, $\delta(p, p_j) > \delta(p, p_k)$ for all $k = 0, 1, \ldots, n-1$, and $k \neq j$. Here $\delta(.,.)$ denotes the Euclidean distance between a pair of points. The furthest point Voronoi diagram $\mathcal{V}(P)$ can be constructed in $O(n \log n)$ time using $O(n)$ space. For any point p in the plane, we can locate the region $\mathcal{R}(p_i)$ containing p in $O(\log n)$ time [10]. If a single point p lies on C', then the distance of p from line L is the maximum among all other points of P. In that case, projection of the point p on the query line L lies in $\mathcal{R}(p)$ and it is the center of the minimum enclosing circle. Let $p_{i_1}, p_{i_2}, \ldots, p_{i_k}$ be the vertices of P lie on the same side of L and they are in clockwise order along the boundary of P. Observe that the sequence of distances between p_{i_j} and line L is unimodal. Hence, using the above data structure we can verify whether C' passes through a single point in $O(\log n)$ time. If C' passes through exactly two points $p_1, p_2 \in P$ then c' is the intersection point of L with the perpendicular bisector of p_1 and p_2. Therefore both p_1 and p_2 are furthest from c'. Observe that, if the circle passes through both p_1 and p_2, then c' must lie on a line-segment e separating $\mathcal{R}(p_1)$ and $\mathcal{R}(p_2)$ where e is an edge of $\mathcal{V}(P)$. From now on we will consider the case where the center c' of the circle C' will be at the point of intersection of L with an edge of $\mathcal{V}(P)$. In the degenerate case, C' may pass through three or more points of P and in that case c' will be a vertex of $\mathcal{V}(P)$. For any point v in $\mathcal{R}(p_k)$, $\rho(v)$ is defined as the distance between v and p_k. Let L intersect the edges $e_1, e_2, \ldots, e_m$ of $\mathcal{V}(P)$ in order at the sequence of intersection points $\{a_1, a_2, \ldots, a_m\}$ respectively. Therefore $\rho(a_i)$ is the radius of the smallest enclosing circle of P with center at a_i. Observe that the function $\rho(\cdot)$ is convex. Note that the center c of the unconstrained smallest enclosing circle C lies on an edge or a vertex of $\mathcal{V}(P)$. Thus, $\mathcal{V}(P)$ may be viewed as a directed tree $\mathcal{T}$ with c as root node, all the Voronoi edges are the branches and all the Voronoi vertices are the internal nodes of $\mathcal{T}$ (see Figure 1(a)). We will use $\pi(v)$ to denote the directed path from c to v in $\mathcal{T}$. Observe that as we go down from c along a path in $\mathcal{T}$, the ρ value of the nodes along that path increases monotonically. If L intersects a path $\pi(v)$ more than once, then the intersection point having minimum depth is the candidate for being c'.

2.2 Preprocessing steps

For any constant $\alpha \geq 0$, let Q be a region such that the distance between any point $q \in Q$, $\rho(q) \leq \alpha$ and for any point $q \in Q^c$, $\rho(q) \geq \alpha$ where Q^c denotes the complement of Q. Then we have the following result.

Lemma 1 *If $\alpha > \rho(c)$ then Q is nonempty convex region containing the point c.*

Proof. Let $C(p_i, \alpha)$ denote the circle centered at p_i ($\in P$) with radius α then $Q = \bigcup_{i=0}^{n-1} \mathcal{R}(p_i) \bigcap C(p_i, \alpha)$. If the circle C is passing through the point $p' \in P$ then c must be in $\mathcal{R}(p') \bigcap C(p', \alpha)$ and hence c is in Q. Observe that the regions $\mathcal{R}(p_i) \bigcap C(p_i, \alpha)$ are either empty or convex for all i and any two regions may share at most a Voronoi edge at the boundary. Here, we omit the proof of convexity of the region Q. □

Note that, the region Q is bounded by circular arcs, and has at most n vertices on edges of Voronoi diagram $\mathcal{V}(P)$. Consider the case, where the query line L intersects Q but does not intersect any Voronoi edge belonging to the region Q. So L must be cut by a circular arc of a circle say $C(p_k, \alpha)$. In this case the center c' must be the projection of the vertex p_k which is the only point to lie on the minimum enclosing circle. In Section 2.1, we have got the solution for the above mentioned case. As we are going to consider only the intersection points of L with the Voronoi edges, we will consider the arcs as straight line segments. Thus the region Q will henceforth be denoted as a convex polygon. Let $\{t_0, t_1, \ldots, t_{n-1}\}$ be the sequence of nodes of $\mathcal{T}$ such that $\rho(t_i) \leq \rho(t_{i+1})$. We store the values $\{\rho(t_i)$, for i=$0, 1, \ldots, n-1\}$ in an array $\mathcal{P}$. So we represent Q_i as a convex polygon such that the distance between each point in Q_i with it's furthest point among point set P is less than or equal to $\mathcal{P}[i]$. A vertex v of Q_i is either a Voronoi vertex or a point on a Voronoi edge of $\mathcal{V}(P)$ where v's are the intersection points of $C(p_j, \mathcal{P}[i])$ and the edges of $\mathcal{R}(p_j)$ for $(0 \leq j \leq n-1)$. If v is on the Voronoi edge associated with $R(p_j)$ and $R(p_k)$, then $d(v, p_j)$=$d(v, p_k)$=$\mathcal{P}[i]$. Joining consecutive vertices of each Q_i in clockwise order with straight lines we get n convex polygons. Note that, $Q_0 \subset Q_1 \subset Q_2 \subset \ldots \subset Q_{n-1}$ as shown in Figure 1. The function $\phi(L)$ maps to an integer m such that the line L intersects all the polygons $Q_m, Q_{m+1}, \cdots, Q_{n-1}$ and does not intersect Q_{m-1}. Note that the length of the radius of the constrained minimum enclosing circle lies between constant $\mathcal{P}[m-1]$ and $\mathcal{P}[m]$. Moreover the center c' lies on an Voronoi edge inside the region $Q_m \setminus Q_{m-1}$. We will use the concept of geometric duality to compute $\phi(L)$.

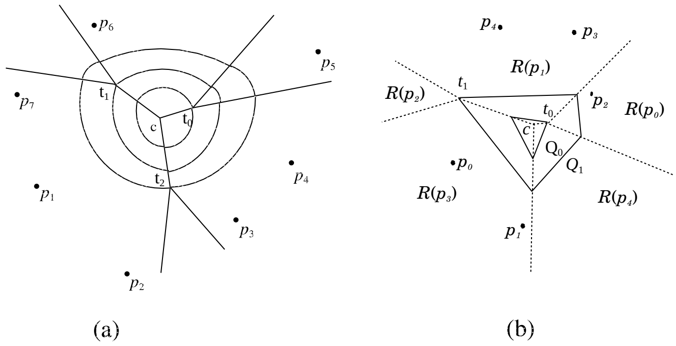

Figure 1: The regions Q_i's defined from $\mathcal{V}(P)$

3 Duality ideas

Consider a non-vertical line $l: \; y = a \cdot x + b$ in a plane where a and b are scalars. Observe that the slope and the intercept (a, b) uniquely define the line l in the primal plane. A simple *duality transform* [1] maps objects from the primal plane to the dual plane. The line l in primal plane is mapped to the point $l' = (a, -b)$ in the dual plane. A point p=(p_x, p_y) in the primal plane is mapped to a line $p': \; y = p_x \cdot x - p_y$ in the dual plane.

Dual mapping is incidence and order preserving: $p \in l$ if and only if $l' \in p'$; and p lies above l if and only if l' lies above p'.

3.1 Dual mapping of a convex polygon

Consider a convex polygon Q with vertices $\{q_0, q_1, \ldots, q_{n-1}\}$ in clockwise order in the primal plane. *Upperhull* of Q can be expressed as a sequence of vertices on the path from the leftmost vertex of Q to the rightmost vertex along the boundary of Q in clockwise direction and hull edges are the line segments joining the adjacent vertices in the sequence. Let $\mathcal{UH}(\mathcal{Q})$ denote the *upperhull* of Q. Similarly $\mathcal{LH}(\mathcal{Q})$ denotes the *lowerhull* of Q and is expressed in similar manner. Let C'_u and C'_l denote the set of the dual lines of the vertices of $\mathcal{UH}(\mathcal{Q})$ and $\mathcal{LH}(\mathcal{Q})$ respectively. $\mathcal{A}(C'_u)$ represent the arrangement of C'_u in dual plane where each element $q'_i \in C'_u$ contributes an edge to the unique bottom cell of arrangement (see Figure 2). This cell is the intersection of the half-planes bounded by the lines in C'_u. The boundary of the bottom cell is the *lower envelope* of the set of lines C'_u and is expressed as ζ'_u. Note that the sequence of vertices of $\mathcal{UH}(\mathcal{Q})$ are in order of increasing x-coordinate. As the slope of q'_i is the x-coordinate of q_i, the clockwise sequence of vertices of $\mathcal{UH}(\mathcal{Q})$ produces an anticlockwise sequences of edges of ζ'_u. Therefore the upper hull of a set of vertices $\mathcal{UH}(\mathcal{Q})$ is same as the *lower envelope* of the set of lines of C_u. By symmetry, lower hull $\mathcal{LH}(\mathcal{P})$ will be mapped to *upper envelope* ζ'_l. Now the chains ζ'_l

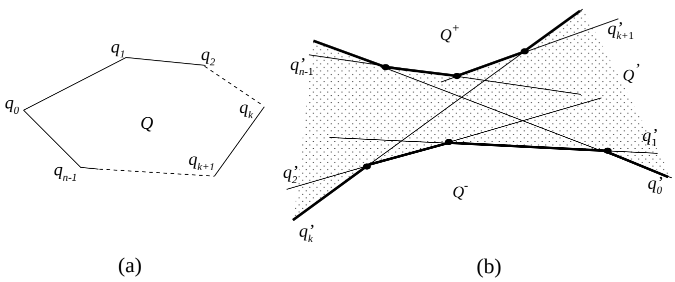

Figure 2: (a) a convex polygon and (b) dual transform of the polygon

and ζ'_u partitions the plane into three regions Q^+, Q' and Q^-. Each point $\alpha \in Q^+$ lies above the chain ζ'_l and each point $\alpha \in Q^-$ lies below the chain ζ'_u (see Figure 2). We denote the region bounded by ζ'_l and ζ'_u as Q'. If l' be any point inside the region bounded by the chains ζ'_l and ζ'_u in the dual plane, l' can be mapped to it's dual line l which intersects the boundary of Q in the primal plane. If line l does not intersect Q, all the vertices of Q are either above l or all the vertices are below the l. It is possible if and only if l' lies outside the region Q'. It is easy to observe that, a point p inside Q maps to line p' inside the region Q' and p' does not intersect ζ'_l or ζ'_u.
The lowerhull and upperhull of convex polygons $Q_0, Q_1, \ldots, Q_{n-1}$ map to the pairs of chains $(\zeta'_{l_0}, \zeta'_{u_0})$, $(\zeta'_{l_1}, \zeta'_{u_1})$, $\ldots$,$(\zeta'_{l_{n-1}}, \zeta'_{u_{n-1}})$ respectively in the dual plane.

Lemma 2 *Let the sets* $\{\zeta'_{l_0}, \zeta'_{l_1}, \ldots, \zeta'_{l_{n-1}}\}$ *and* $\{\zeta'_{u_0}, \zeta'_{u_1}, \ldots, \zeta'_{u_{n-1}}\}$ *be denoted by* Δ_1 *and* Δ_2 *respectively. Then,*

(i) any pair of chains from the set $\Delta_1 \cup \Delta_2$ *are non-intersecting,*

(ii) each chain in the set Δ_1 *lies above* Δ_2*,*

(iii) for any $i > j$, ζ'_{l_i} *lies above* ζ'_{l_j} *and similarly* ζ'_{u_i} *lies below* ζ'_{u_j}*.*

Proof. Let two chains ζ'_{l_i} and ζ'_{l_j} for $(i < j)$ have a common intersection point. Then we can always detect two points α and β in the dual plane such that α lies above ζ'_{l_i} but below ζ'_{l_j} and β lies above ζ'_{l_j} and below ζ'_{l_i}. Then the corresponding line α' in the primal plane intersect Q_j but does not intersect Q_i. Similarly β' intersect Q_i but does not intersect Q_j. This contradicts the containment relationship of these two polygons. As Q_i is entirely contained in Q_j the two envelopes cannot cross. Rest part of the proof is omitted here. □

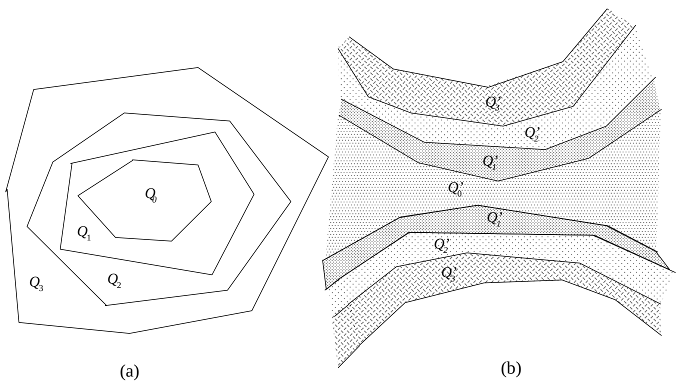

Figure 3: (a) a set of concentric convex polygons and (b) their dual transformation

Now the interior region of the convex polygons $Q_0, Q_1, \ldots, Q_{n-1}$ in the primal plane are mapped to the region bounded by the pair of chains $(\zeta'_{l_0}, \zeta'_{u_0})$, $(\zeta'_{l_1}, \zeta'_{u_1})$, $\ldots$, $(\zeta'_{l_{n-1}}, \zeta'_{u_{n-1}})$ respectively in the dual plane. From Lemma 2, $Q'_0 \subset Q'_1 \subset Q'_2 \subset \ldots \subset Q'_{n-1}$.

Lemma 3 *If* $\phi(L) = i$*, the point* L' *in the dual plane corresponding to the line* L*, will be located in* $Q'_i \setminus Q'_{i-1}$*.*

The number of vertices in each polygon Q_i is at most n. The polygon Q_i can be constructed by identifying all the points in Voronoi edges which are $\mathcal{P}[i]$ distance apart from its furthest point among P. The convex hull of these points gives the polygon Q_i replacing the boundary arcs by line segments. There can be at most n convex polygons and the number of vertices in each convex polygon is at most $n-2$.

Lemma 4 *All the convex polygons and their image in the dual plane can be constructed in $O(n^2)$ time and space.*

4 Algorithm for answering Query

In the preprocessing phase we have constructed a set of concentric convex polygons $\{Q_0, Q_1, \ldots, Q_{n-1}\}$. Using duality principles we map each Q_i to it's dual image $\{\zeta'_{l_i}, \zeta'_{u_i}\}$ in the dual plane where the region bounded by $\{\zeta'_{l_i}, \zeta'_{u_i}\}$ is Q'_i. Our algorithm for finding c' proceeds in two phases as follows:

- **Phase 1:** Compute $\phi(L)$ that is, locate the region Q'_k which contains the dual point L' and L' does not lie inside Q'_{k-1}.
- **Phase 2:** Find the intersection points of L and the edges of $\mathcal{V}(P)$ which lie inside Q_k. The intersection point whose distance is minimum from it's furthest point is the candidate for c'.

We implement Phase 1 as a point-location query in a given two-dimensional monotone subdivision [5]. As all the chains are x-monotone the sequence of chains $\{\zeta'_{l_{n-1}}, \zeta'_{l_{n-2}}, \ldots, \zeta'_{l_0}, \zeta'_{u_0}, \zeta'_{u_1}, \ldots, \zeta'_{u_{n-1}}\}$ partitions the plane into $2n+1$ disjoint monotone regions. Let the above set of chains be denoted as $\mathcal{S}$ whose elements are renamed as $\{s_1, s_2, \ldots, s_{2n}\}$ and partitions the plane into a set of regions $\Re = \{r_0, r_1, \ldots, r_{2n}\}$, where $r_i \cup r_{2n-i}$ denotes the region $Q'_{n-i} \setminus Q'_{n-i-1}$, for $i = 1, 2, \ldots, n-1$. We say a region r_i is below r_j and write $r_i \ll r_j$ for $i < j$, if $i \neq j$ and for any two points $p=(\pi_1, \pi_2)$ and $q=(\pi_1, \psi_2)$ on r_i and r_j respectively, we have $\psi_2 \leq \pi_2$. So we have
$r_0 \ll s_1 \ll r_1 \ll s_2 \ll r_2 \ldots \ll s_{2n} \ll r_{2n}$.
Therefore, in the monotone subdivision the regions are vertically ordered. A balanced binary search tree data structure can be constructed using $O(n^2)$ time and space similar to the method described in [5], where each node v stores a layered dag that is a list obtained by merging a chain from $\mathcal{S}$ representing that node along with layered dags of its children. Using the structure, $\phi(L)$ can be computed in $O(\log n)$ time.

Now, we can locate the points of intersection between $Q_{\phi(L)}$ and the line L in $O(\log n)$ time. The vertices of $Q_{\phi(L)}$ is partitioned into two group say, G_1 and G_2 depending on their position on the (left and right) side of L. Observe that, all the edges from $e_1, e_2, \ldots, e_n$ of $\mathcal{V}(P)$ which are intersected by L in the region $Q_{\phi(L)}$ must pass through the vertices of same group, that is either the vertices in G_1 or G_2 but not both. Again, the sequence of ρ values at the ordered set of intersection points is unimodal. Therefore, in $O(\log n)$ search we can identify the intersection point having minimum ρ value and consequently the center of the minimum enclosing circle c' on L.

Theorem 5 *Given a point set $P = \{p_1, p_2, \ldots, p_n\}$ and a query line L, the minimum enclosing circle having center on L can be reported in $O(\log n)$ time. Both the preprocessing time and space complexities are $O(n^2)$.*

In case L is a line segment, then we first locate the center c' on the line containing L. If c' is not on the line segment L, then one of the end points of L nearer to c' is the solution.

References

[1] M. de Berg, M. van Kreveld, M. Overmars, and O. Schwarzkopf, *Computational Geometry - Algorithms and Applications*, Springer-Verlag, 1997.

[2] P. Bose and G. Toussaint, *Computing the constrained Euclidean, geodesic and link center of a simple polygon with applications*, Proc. of Pacific Graphics International, pp. 102-112, 1996.

[3] P. Bose and Q. Wang, *Facility location constrained to a polygonal domain*, Theoretical Informatics, 5th Latin American Symposium, LNCS 2286, pp. 153-164, 2002.

[4] J. Elzinga and D. W. Hearn, *Geometrical solutions to some minimax location problems*, Transp. Sci., vol. 6, pp. 379-394, 1972.

[5] H. Edelsbrunner, L. J. Guibas and J. Stolfi, *Optimal point location in a monotone subdivision*, SIAM Journal on Computing, pp. 317 - 340, 1986.

[6] F. Hurtado, V. Sacristan and G. Toussaint, *Facility location problems with constraints*, Studies in Locational Analysis, vol. 15, pp. 17-35, 2000.

[7] D.T. Lee, *Furthest neighbour Voronoi diagrams and applications*, Report80-11-FC-04, Dept. Elect. Engrg. Comput. Sci., Northwestern Univ., Evanston, IL, 1980.

[8] N. Megiddo, *Linear-time algorithms for linear programming in R^3 and related problems*, SIAM J. Comput., vol. 12, pp. 759-776, 1983.

[9] F. P. Preparata, *Minimum spanning circle*, Technical report, Univ. Illinois, Urbana, IL, in Steps into Computational Geometry, 1977.

[10] F. P. Preparata and M. I. Samos, *Computational Geometry: An Introduction, Second edition,* Springer Verlag, 1990.

[11] S. Roy, A. Karmakar, S. Das and S. C. Nandy, *Constrained Minimum Enclosing Circle with Center on a Query Line Segment*, Proc. Mathematical Foundation of Computer Science, pp. 765-776, 2006.

[12] M.I. Shamos and D. Hoey, *Closest-point problem*, Proc. 16th Annual IEEE Sympos. Found. Comput. Sci., pp. 151-162, 1975.

[13] J. J. Sylvester, *A question in the geometry of situation*, Quarterly Journal of Mathematices, pp. 1-79, 1857.