Isometric Morphing of Triangular Meshes *

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Abstract

We present a novel approach to morph between two isometric poses of the same non-rigid object given as triangular meshes. We model the morphs as linear interpolations in a suitable shape space $\mathcal{S}$. For triangulated $3D$ polygons, we prove that interpolating linearly in this shape space corresponds to the most isometric morph in $\mathbb{R}^3$. We extend this shape space to arbitrary triangulations in $3D$ using a heuristic approach.

1 Introduction

Given two isometric poses of the same non-rigid object as triangular meshes $S^{(0)}$ and $S^{(1)}$ with known point-to-point correspondences, we aim to find a smooth isometric deformation between the poses. Interpolating smoothly between two given poses is called *morphing*. We achieve this by finding shortest paths in a shape space similar to the approach by Kilian et al. [5]. We propose a novel shape space.

A deformation of a shape represented by a triangular mesh is isometric if and only if all triangle edge lengths are preserved during the deformation [5]. We call a morph $S^{(t)}, 0 < t < 1$ between two (possibly nonisometric) shapes $S^{(0)}$ and $S^{(1)}$ *most isometric* if it minimizes the sum of the absolute values of the differences between the corresponding edge lengths of two consecutive shapes summed over all shapes $S^{(t)}$, for t in $[0, 1]$[1]. In this paper, we examine isometric morphs of general *triangular manifold meshes* in $3D$ and of *triangulated* $3D$ *polygons*, which are triangular meshes with no interior vertices. We introduce a new shape space $\mathcal{S}$ for triangulated $3D$ polygons that has the property that interpolating linearly in shape space corresponds to the most isometric morph in $\mathbb{R}^3$. We then extend this shape space to arbitrary triangulations in $3D$ using a heuristic approach. Note that self-intersections may occur when morphing.

Computing a smooth morph from one pose of a shape in two or three dimensions to another pose of the same shape has numerous applications. For example in computer graphics and computer animation this problem has received considerable attention [7, 1].

Recently, Kilian et al. [5] used shape space representations to guide morphs and other more general deformations between shapes represented as triangular meshes. Each shape is represented by a point in a high-dimensional shape space and deformations are modeled as geodesics in shape space. The geodesic paths in shape space are found using an energy-minimization approach. Before Kilian et al. [5] presented the use of a shape space for shape deformation and exploration of triangular meshes, shape space representations were developed to deform shapes in different representations. Cheng et al. [2] proposed an approach that deforms shapes given in skin representation, which is a union of spheres that are connected via blending patches of hyperboloids, with the help of a suitable shape space. Furthermore, algorithms for deforming curves with the help of shape space representations were proposed by Younes [10] and Klassen et al. [6]. Eckstein et al. [3] propose a generalized gradient descent method similar to the approach by Kilian et al. that can be applied to deform triangular meshes. All of these approaches depend on solving a highly non-linear optimization problem with many unknown variables using numerical solvers. It is therefore not guaranteed that the globally optimal solution is found.

2 Theory of Shape Space for Triangulated $3D$ Polygons

This section introduces a novel shape space for triangulated $3D$ polygons with the property that interpolating linearly in shape space corresponds to the most isometric morph in $\mathbb{R}^3$. The dimensionality of the shape space is linear in the number of vertices of the deformed polygon.

We start with two triangulated $3D$ polygons $P^{(0)}$ and $P^{(1)}$ corresponding to two almost isometric poses of the same non-rigid object. We assume that the point-to-point correspondence of the vertices $P^{(0)}$ and $P^{(1)}$ are known. Furthermore, we assume that both $P^{(0)}$ and $P^{(1)}$ share the same underlying mesh structure M. Hence, we know the mesh structure M with two sets of ordered vertex coordinates $V^{(0)}$ and $V^{(1)}$ in $\mathbb{R}^3$, where M is an outer-planar graph. We will show that we can represent $P^{(0)}$ and $P^{(1)}$ as points $p^{(0)}$ and $p^{(1)}$ in a shape space $\mathcal{S}$, such that each point $p^{(t)}$ that is a linear interpolation between $p^{(0)}$ and $p^{(1)}$ corresponds to a triangular mesh $P^{(t)}$ isometric to $P^{(0)}$ and $P^{(1)}$ in $\mathbb{R}^3$.

Let M consist of n vertices. As M is a triangulation of a $3D$ polygon with n vertices, M has $2n - 3$ edges and $n - 2$

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[1] Kilian et al. [5] consider squared edge differences.

triangles. We assign an arbitrary but fixed order on the vertices, edges, and faces of M. We eliminate rigid transformations by positioning $P^{(0)}$ and $P^{(1)}$ such that the first vertex v is incident to the origin, the first edge e of M incident to v is aligned along the positive x-axis, and the first triangle containing e lies on the x, y plane. The shape space $\mathcal{S}$ is defined as follows. The first $2n-3$ coordinates are the lengths of the edges in M in order. The final $2(n-2)$ coordinates are the outer normal directions of the triangles in M in spherical coordinates, in order. Hence, the shape space $\mathcal{S}$ has dimension $2n-3+2(n-2) = 4n-7 = \Theta(n)$.

In the following, we prove that interpolating linearly between $P^{(0)}$ and $P^{(1)}$ in shape space yields the most isometric morph. To interpolate linearly in shape space, we interpolate the edge lengths by a simple linear interpolation. That is, $p_k^{(t)} = tp_k^{(0)} + (1-t)p_k^{(1)}$, where $p_k^{(x)}$ is the kth coordinate of $p^{(x)}$. The normal vectors are interpolated using geometric spherical linear interpolation (SLERP) [8]. That is, $p_k^{(t)} = \frac{\sin(1-t)\Theta}{\sin\Theta} p_k^{(0)} + \frac{\sin t\Theta}{\sin\Theta} p_k^{(1)}$, where Θ is the angle between the two directions that are interpolated.

To study interpolation in shape space, we make use of the *dual graph* $D(M)$ of M. The dual graph $D(M)$ has a node for each triangle of M. We denote the dual node corresponding to face f of M by $D(f)$. Two nodes of $D(M)$ are joined by an arc if the two corresponding triangles in M share an edge. We denote the dual arc corresponding to an edge e of M by $D(e)$. Note that because M meshes a $3D$ polygon, it is an outer-planar triangular graph and so the dual graph of M is a binary tree. An example of a mesh M with its dual graph $D(M)$ is shown in Figure 1.

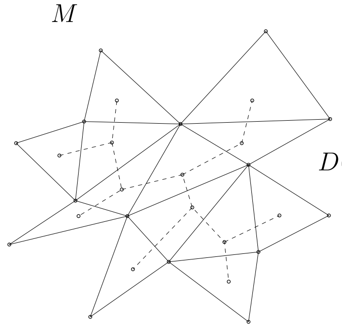


Figure 1: *A mesh M with its dual graph $D(M)$.*

Theorem 1 *Let M be the underlying mesh structure of the triangulated $3D$ polygons $P^{(0)}$ and $P^{(1)}$. The linear interpolation $p^{(t)}$ between $p^{(0)}$ and $p^{(1)}$ in shape space $\mathcal{S}$ for $0 \leq t \leq 1$ has the following properties:*

1. *The mesh $P^{(t)} \in \mathbb{R}^3$ that corresponds to $p^{(t)} \in \mathcal{S}$ is uniquely defined and has the underlying mesh structure M. We can compute this mesh using a traversal of the binary tree $D(M)$ in $\Theta(n)$ time.*

2. *If $P^{(0)}$ and $P^{(1)}$ are isometric, then $P^{(t)}$ is isometric to $P^{(0)}$ and $P^{(1)}$. If $P^{(0)}$ and $P^{(1)}$ are not isometric, then each edge length of $P^{(t)}$ linearly interpolates between the corresponding edge lengths of $P^{(0)}$ and $P^{(1)}$.*

3. *The coordinates of the vertices of $P^{(t)}$ are a continuous function of t.*

Due to page restrictions, the proof of this theorem is omitted in this abstract, but can be found in [9]. Note that Theorem 1 implies that the most isometric morph is found as all edge lengths are linearly interpolated.

Because $D(M)$ has complexity $\Theta(n)$, we can traverse $D(M)$ in $\Theta(n)$ time. Hence, we can compute intermediate deformation poses in $\Theta(n)$ time each.

Using the proposed algorithm, we deform the polygon shown in Figure 3 (a) to the polygon shown in Figure 3 (i). The morph is illustrated in Figures 3 (b)-(h). All of the intermediate poses are isometric to the start and end poses. The overlayed poses are shown in Figure 2.

Figure 2: *Most isometric morph of a simple polygon. The start polygon is a $3D$ polygon obtained by discretizing the curve $y = \sin(x)$ and by adding thickness to the curve along the z-direction. The end polygon is similarly obtained from $y = -\sin(x)$.*

3 Generalization to Triangular Meshes

This section extends the shape space from the previous section to arbitrary connected triangular meshes. We can no longer guarantee the properties of Theorem 1, because the dual graph of the triangular mesh M is no longer a tree.

Given two triangular meshes $S^{(0)}$ and $S^{(1)}$ corresponding to two almost isometric poses of the same non-rigid object with known point-to-point correspondence, we know one mesh structure M with two sets of ordered vertex coordinates $V^{(0)}$ and $V^{(1)}$ in $\mathbb{R}^3$. As before, we can represent $S^{(0)}$ and $S^{(1)}$ as points $s^{(0)}$ and $s^{(1)}$ in a shape space $\mathcal{S}$ using the same shape space points as in Section 2. Let $s^{(t)}$ be the linear interpolation of $s^{(0)}$ and $s^{(1)}$ in $\mathcal{S}$, where the linear interpolation is computed as outlined in Section 2. The existence of a mesh $S^{(t)} \in \mathbb{R}^3$ that has the underlying mesh structure M and that corresponds to $s^{(t)}$ is no longer guaranteed. This can be seen using the example shown in Figure 4. Figure 4(a) and (b) show two isometric meshes $S^{(0)}$ and $S^{(1)}$. The dual graph $D(M)$ of the mesh structure M is a simple cycle. Note that although the start and the end pose are isometric, we cannot find an intermediate pose that satisfies

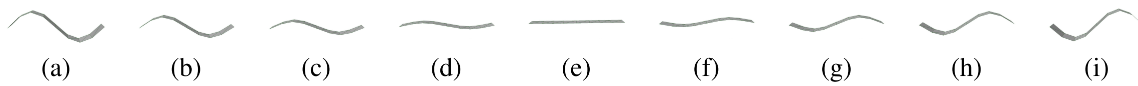


Figure 3: *Most isometric morph of a simple polygon from pose (a) to pose (i) obtained using the polygon algorithm.*

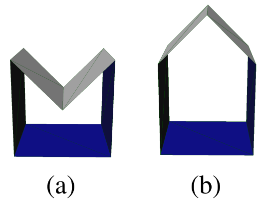


Figure 4: *Example of isometric triangular meshes where most isometric intermediate poses do not exist.*

all of the interpolated normal vectors and that is isometric to $S^{(0)}$ and $S^{(1)}$.

Let M consist of n vertices. As M is a planar graph, M has $\Theta(n)$ edges and $\Theta(n)$ triangles. The shape space $\mathcal{S}$ is defined using the same shape space points as in Section 2. The shape space $\mathcal{S}$ has dimension $\Theta(n)$. As before, we interpolate linearly in shape space by interpolating the edge lengths by a simple linear interpolation. The following observation is illustrated in Figure 4.

Observation 1 *Given a triangular mesh $S^{(t)}$ with underlying mesh structure M, point $s^{(t)}$ in $\mathcal{S}$ is uniquely determined. However, the inverse operation, that is computing a triangular mesh $S^{(t)}$ given a point $s^{(t)} \in \mathcal{S}$, is ill-defined.*

To compute a unique triangular mesh $S^{(t)}$ given a point $s^{(t)} \in \mathcal{S}$ that linearly interpolates between $s^{(0)}$ and $s^{(1)}$, such that $S^{(t)}$ represents the information given in $s^{(t)}$ well, we use the dual graph $D(M)$ of M. Unlike in Section 2, $D(M)$ is not necessarily a tree. Our algorithm therefore operates on a minimum spanning tree $T(M)$ of $D(M)$. The tree $T(M)$ is computed by assigning a weight to each arc e of $D(M)$. The weight of e is equal to the difference in dihedral angle of the supporting planes of the two triangles of M corresponding to the two endpoints of e. That is, we compute the dihedral angle between the two supporting planes of the two triangles of M corresponding to the two endpoints of e for the start pose $S^{(0)}$ and for the end pose $S^{(1)}$. The weight of e is then set as the difference between those two dihedral angles, which corresponds to the change in dihedral angle during the deformation. The weight can therefore be seen as a measure of rigidity. The smaller the weight, the smaller the change in dihedral angle between the two triangles during the deformation, and the more rigidly the two triangles move with respect to each other. As $T(M)$ is a minimum spanning tree, $T(M)$ contains the arcs corresponding to the most rigid components of M.

We compute $S^{(t)}$ by traversing $T(M)$. However, unlike in Section 2, setting the vertex coordinates of a vertex v of $S^{(t)}$ using two paths from the root of $T(M)$ to two triangles containing v can yield two different resulting coordinates for v. An example of this situation is given in Figure 5, where the coordinates of v can be set by starting at $root(T(M))$, and traversing the arcs e_2 and e_3 of $T(M)$ or by traversing the arcs e_1, e_4, and e_5 of M. We call the different coordinates computed for v in $T(M)$ *candidate coordinates* of v. Our algorithm computes the coordinates of each vertex $v \in S^{(t)}$ as the average of all the candidate coordinates of v.

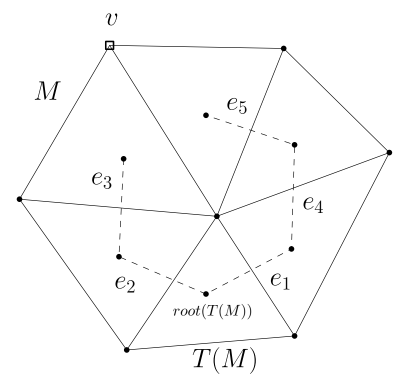
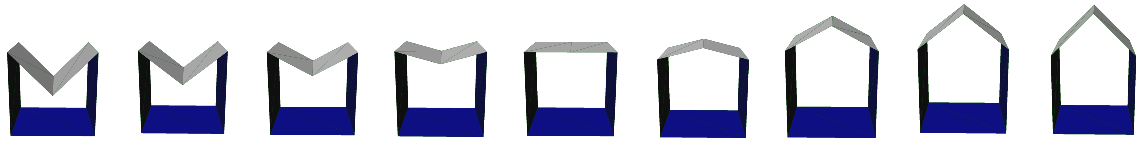


Figure 5: *A mesh M with its dual tree $T(M)$.*

To analyze the maximum number of candidate coordinates that can occur for a vertex in $S^{(t)}$, let e denote an edge of M such that $D(e)$ is in $T(M)$. Let v denote the vertex of $S^{(t)}$ opposite e in the triangle corresponding to an endpoint of $D(e)$, such that the coordinates of v are computed when traversing $D(e)$. This is illustrated in Figure 6. Let d_1 and d_2 denote the total number of candidate coordinates of the two endpoints of e. As we compute $d_1 d_2$ candidate coordinates for v by traversing $D(e)$, we can bound the number of candidate coordinates of v computed using the path through $D(e)$ by $d_1 d_2$. The number of candidate coordinates for the two endpoints of the edge corresponding to the first edge is one. Furthermore, each vertex v can be reached by at most $deg(v)$ paths in $T(M)$, where $deg(v)$ denotes the degree of vertex v in M. As each path in $T(M)$ has length at most $m-1$, where $m = O(n)$ is the number of triangles of M, we can bound the total number of candidate coordinates in $S^{(t)}$ by $\sum_{v \in V} 2^{m-1} deg(v) = 2n2^{m-1}$, where V is the vertex set.

Our algorithm finds a triangular mesh $S^{(t)}$ corresponding to $s^{(t)}$ that is isometric to $S^{(0)}$ and $S^{(1)}$ if such a mesh exists, because all of the candidate coordinates are equal in this case and taking their average yields the desired result. If there is no isometric mesh corresponding to $s^{(t)}$, our algorithm finds the *nearly isometric morph* as a unique mesh that weighs all the evidence given by $T(M)$ equally. By choosing $T(M)$ as a minimum spanning tree based on weights repre-



(a) (b) (c) (d) (e) (f) (g) (h) (i)

Figure 7: *Nearly isometric morph of a cycle from pose (a) to pose (i) obtained using the exponential algorithm.*

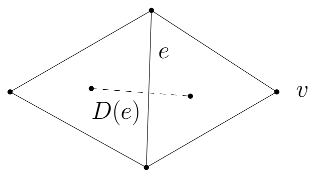


Figure 6: *Illustration of how to bound the number of candidate coordinates of v computed using the path through $D(e)$.*

senting rigidity, we allocate rigid parts of the model more emphasis than non-rigid parts. The reason for this is that in most morphs, triangles close to non-rigid joints are deformed more than triangles in mainly rigid parts of the model. We conclude with the following.

Proposition 2 *Let $S^{(0)}$ and $S^{(1)}$ denote two isometric connected triangular meshes and let $s^{(0)}$ and $s^{(1)}$ denote the corresponding shape space points, respectively. We can compute a unique triangular mesh $S^{(t)}$ representing the information given in the linear interpolation $s^{(t)}, 0 \leq t \leq 1$ of $s^{(0)}$ and $s^{(1)}$, in exponential time. We find a triangular mesh $S^{(t)}$ corresponding to $s^{(t)}$ that is isometric to $S^{(0)}$ and $S^{(1)}$ if such a mesh exists.*

The algorithm can easily be extended to work for a non-connected triangular mesh M by removing rigid transformations for each connected component of M using local coordinate systems. We can then adapt the algorithm by finding the dual graph $D(M)$ and a minimum spanning tree $T(M)$ for each connected component of M. With this information, we can traverse the graph as described above.

Using the proposed algorithm, we deform the model shown in Figure 7. We aim to smoothly and isometrically deform the pose shown in Figure 7(a) to the pose shown in Figure 7(i). As mentioned previously, there is no isometric deformation between the poses that interpolates the triangle normals. The result of our algorithm is shown in Figures 7(b)-(h). Note that all triangle normals are interpolated and the symmetry of the model is preserved. Furthermore, all edge lengths with the exception of the edges of the four top faces are interpolated.

4 Conclusion

We presented a novel approach to morph efficiently between isometric poses of triangular meshes in a novel shape space. The main advantage of this morphing method is that the most isometric morph is always found in linear time when triangulated $3D$ polygons are considered. For general triangular meshes, the approach cannot be proven to find the optimal solution. However, this paper presents a heuristic approach to find a morph for general triangular meshes. More efficient heuristics are explored in the accompanying technical report [9].

A direction for future work is to find an efficient way of morphing triangular meshes while guaranteeing that no self-intersections occur. For polygons in two dimensions, this problem was solved using an approach based on energy minimization [4].

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