Monochromatic simplices of any volume

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Abstract

We give a very short proof of the following result of Graham from 1980: For any finite coloring of $\mathbb{R}^d$, $d \geq 2$, and for any $\alpha > 0$, there is a monochromatic $(d+1)$-tuple that spans a simplex of volume α. Our proof also yields new estimates on the number $A = A(r)$ defined as the minimum positive value A such that, in any r-coloring of the grid points $\mathbb{Z}^2$ of the plane, there is a monochromatic triangle of area exactly A.

1 Introduction

The classical theorem of Van der Waerden states that if the set of integers $\mathbb{Z}$ is partitioned into two classes then at least one of the classes must contain an arbitrarily long arithmetic progression [20]. The result holds for any fixed number of classes [17]. Let $W(k, r)$ denote the Van der Waerden numbers: $W = W(k, r)$ is the least integer such that for any r-coloring of $[1, W]$, there is a monochromatic arithmetic progression of length k. The following generalization of Van der Waerden's theorem to two dimensions is given by Gallai's theorem [17]: If the grid points $\mathbb{Z}^2$ of the plane are finitely colored, then for any $t \in \mathbb{N}$, there exist $x_0, y_0, h \in \mathbb{Z}$ such that the t^2 points $\{(x_0 + ih, y_0 + jh) \mid 0 \leq i, j \leq t-1\}$ are of the same color.

Many extensions of these Ramsey type problems to the Euclidean space have been investigated in a series of papers by Erdős et al. [9, 10, 11] in the early 1970s, and by Graham [12, 13, 14]. See also Ch. 6.3 in the problem collection by Braß, Moser and Pach [4], and the recent survey articles by Braß and Pach [3] and by Graham [15, 16]. For a related coloring problem on the integer grid, see [6].

In 1980, answering a question of Gurevich, Graham [12] proved that for any finite coloring of the plane, and for any $\alpha > 0$, there is a monochromatic triangle of area α. In their survey article, Braß and Pach [3] observed that for any 2-coloring of the plane there is a monochromatic triple that spans a triangle of unit area, and asked whether this holds for any finite coloring, apparently unaware of Graham's solution [12]. This also brought the problem to our attention. Graham's proof was quite involved, and was later simplified by Adhikari [1] using the same main idea. Adhikari and Rath [2] have subsequently obtained a similar result for trapezoids. See also [8] for discussions on this and other related problems. Here we present a very short proof of Graham's result [12] in the following theorem, which gives new insight into the problem and also has quantitative implications (see Theorem 4).

Theorem 1 (Graham [12]). *For any finite coloring of the plane, and for any $\alpha > 0$, there is a monochromatic triangle of area α.*

As a corollary of the planar result, one obtains a similar result concerning simplices in d-space for all $d \geq 2$. This was pointed out by Graham [12] without giving details. For completeness, we include our short proof of the following theorem.

Theorem 2 (Graham [12]). *Let $d \geq 2$. For any finite coloring of $\mathbb{R}^d$, and for any $\alpha > 0$, there is a monochromatic $(d+1)$-tuple that spans a simplex of volume α.*

Using a general "product" theorem for Ramsey sets [12, Theorem 3], Graham extended Theorem 2 to the following much stronger result that accommodates all values of α in the same color class. Theorem 3 below can also be obtained using the same "product" theorem in conjunction with our short proof of Theorem 2.

Theorem 3 (Graham [12]). *Let $d \geq 2$. For any finite coloring of $\mathbb{R}^d$, some color class has the property that, for any $\alpha > 0$, it contains a monochromatic $(d+1)$-tuple that spans a simplex of volume α.*

Let $r \geq 2$. Graham [12] defined the number $T = T(r)$ as the minimum value $T > 0$ such that, in any r-coloring of the grid points $\mathbb{Z}^2$ of the plane, there is a monochromatic *right* triangle of area exactly T. We now define $A(r)$ for *arbitrary* triangles. Let $A = A(r)$ be the minimum value $A > 0$ such that, in any r-coloring of the grid points $\mathbb{Z}^2$ of the plane, there is a monochromatic grid triangle of area exactly A. Graham's proof of Theorem 1 [12] shows that $T(r)$ exists, which obviously implies the existence of $A(r)$. We clearly have $A(r) \leq T(r)$.

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Graham [12] obtained an upper bound $T(r) \le \widehat{T}(r) = S_1 \cdot S_2 \cdots S_r$, where

$$S_1 = 1, \quad S_{i+1} = (S_i+1)! \cdot W(2(S_i+1)!+1, i+1)!\,.$$

In Theorem 4 below, we derive an upper bound $A(r) \le \widehat{A}(r)$, and show that $\widehat{A}(r) \ll \widehat{T}(r)$. While Graham [12] finds a *right* monochromatic grid triangle of area exactly $\widehat{T}(r)$, we find an *arbitrary* monochromatic grid triangle of area exactly $\widehat{A}(r)$. However, as far as we are concerned in answering the original question of Gurevich, or the question of Braß and Pach [3], this aspect is irrelevant.

For the lower bound, we clearly have $A(r) \ge 1/2$ because the triangles are spanned by grid points. Let l.c.m. denote the least common multiple of a set of numbers. Graham [12] notes the following lower bound for $T(r)$ based on cyclic colorings of $\mathbb{Z}^2$ (without giving details):

$$T(r) \ge \frac{1}{2} \times \text{l.c.m.}\,(2,3,\ldots,r) = e^{(1+o(1))r}.$$

We will show that the same lower bound holds for $A(r)$ as well.

Theorem 4 *Let $A = A(r)$ be the minimum value $A > 0$ such that, in any r-coloring of the grid points $\mathbb{Z}^2$ of the plane, there is a monochromatic triangle of area exactly A. Let*

$$H = \left\lfloor \frac{W(r!+1, 2^r-1)-1}{r!} \right\rfloor, \text{ and } \widehat{A}(r) = H! \cdot r!.$$

Then $\frac{1}{2} \times$ l.c.m. $(2,3,\ldots,r) \le A(r) \le \widehat{A}(r)$, where $\widehat{A}(r) \ll \widehat{T}(r)$ for sufficiently large r.

It is worth noting the connection between the problems we discussed here and the following old and probably difficult problem of Erdős [7, 8]: Does there exist an absolute constant B such that any measurable plane set E of area B contains the vertices of a unit-area triangle? The answer is known only in certain special cases: if E has infinite area, or even if E has positive area but is unbounded, then E has the desired property; see [5, Problem G13, pp. 182] and [19]. It follows that if in a finite coloring of the plane each color class is measurable, then the largest color class, say E, has infinite area, and hence there is a monochromatic triple that spans a triangle of unit area. But of course, this case is already covered by Theorem 1.

2 Proof of Theorem 1

Let $R = \{1,2,\ldots,r\}$ be the set of colors. Pick a Cartesian coordinate system (x,y). Consider the finite coloring of the lines induced by the coloring of the points: each line is colored (labeled) by the subset of colors $R' \subseteq R$ used in coloring its points. Note that this is a (2^r-1)-coloring of the lines.

Set $N = W(r!+1, 2^r-1)$. By Van der Waerden's theorem, any (2^r-1)-coloring of the N horizontal lines $y = i$, $i = 0,1,\ldots,N-1$, contains a monochromatic arithmetic progression of length $r!+1$: $y_0, y_0+k, \ldots, y_0+r!k$. Let $\mathcal{L} = \{\ell_i \mid 0 \le i \le r!\}$, where $\ell_i : y = y_0 + ik$ for some integers $y_0 \ge 0$, $k \ge 1$. Each of these lines is colored by the same set of colors, say $R' \subseteq R$.

Set $x = 2\alpha/r!k$. Consider the $r+1$ points of ℓ_0 with x-coordinates $0, x, \ldots, rx$. By the pigeon-hole principle, two of these points, say a and b, share the same color, and their distance is jx for some $j \in R$. Pick any point c of the same color on the line $\ell_{r!/j}$ (note that $r!/j$ is a valid integer index, and this is possible by construction!). The three points a, b, c span a monochromatic triangle Δabc of area

$$\frac{1}{2} \cdot jx \cdot \frac{r!k}{j} = \frac{1}{2} \cdot \frac{2j\alpha}{r!k} \cdot \frac{r!k}{j} = \alpha,$$

as required.

3 Proof of Theorem 2

We proceed by induction on d. The basis $d = 2$ is verified in Theorem 1. Let now $d \ge 3$. Assume that the statement holds for dimension $d-1$, and we prove it for dimension d.

Consider the finite coloring of the hyperplanes induced by the coloring of the points by a set R of r colors: each hyperplane is colored (labeled) by the subset of colors $R' \subseteq R$ used in coloring its points. We thus get a (2^r-1)-coloring of the hyperplanes. Pick a Cartesian coordinate system $(x_1,\ldots,x_d)$, and consider the set of parallel hyperplanes $x_d = i$, $i \in \mathbb{N}$. Let π_1 and π_2 be two parallel hyperplanes colored by the same set of colors, say $R' \subseteq R$. Let h be the distance between π_1 and π_2. By induction, π_1 has a monochromatic d-tuple that spans a simplex of volume $\alpha d/h$. Pick a point of the same color in π_2, and note that together they form a $(d+1)$-tuple that spans a simplex of volume

$$\frac{1}{d} \cdot \frac{\alpha d}{h} \cdot h = \alpha,$$

as required.

4 Proof of Theorem 4

We note that our short proof of Theorem 1 does not imply the existence of $A(r)$, since the triangle found there is not necessarily a *grid* triangle. We proceed as in the proof of Theorem 1, but with different settings for the parameters. Set $\alpha = \widehat{A}(r)$. We will show that there is a grid triangle of area exactly α.

Let $R = \{1, 2, \dots, r\}$ be the set of colors. Pick a Cartesian coordinate system (x, y). Consider the finite coloring of the lines induced by the coloring of the *grid* points on the lines: each line is colored (labeled) by the subset of colors $R' \subseteq R$ used in coloring its grid points. Note that this is a $(2^r - 1)$-coloring of the lines.

Set $N = W(r! + 1, 2^r - 1)$. By Van der Waerden's theorem, any $(2^r - 1)$-coloring of the N horizontal grid lines $y = i$, $i = 0, 1, \dots, N - 1$, contains a monochromatic arithmetic progression of length $r! + 1$: $y_0, y_0 + k, \dots, y_0 + r!k$. Let $\mathcal{L} = \{\ell_i \mid 0 \le i \le r!\}$, where $\ell_i : y = y_0 + ik$ for some integers $y_0 \ge 0$, $k \ge 1$. Each of these grid lines is colored by the same set of colors, say $R' \subseteq R$. The common difference of this arithmetic progression is

$$k \le \left\lfloor \frac{W(r! + 1, 2^r - 1) - 1}{r!} \right\rfloor = H.$$

Set $x = 2\alpha/r!k$. Since $\alpha = \widehat{A}(r) = H! \cdot r!$, we have $x = 2H!/k \in \mathbb{N}$. Consider the $r + 1$ grid points on ℓ_0 with x-coordinates $0, x, \dots, rx$. By the pigeon-hole principle, two of these points, say a and b, share the same color, and their distance is jx for some $j \in R$. Pick any grid point c of the same color on the line $\ell_{r!/j}$ (note that $r!/j$ is a valid integer index, and this is possible by construction!). The three grid points a, b, c span a monochromatic triangle Δabc of area

$$\frac{1}{2} \cdot jx \cdot \frac{r!k}{j} = \frac{1}{2} \cdot \frac{2j\alpha}{r!k} \cdot \frac{r!k}{j} = \alpha,$$

as required. This completes the proof of the existence of $A(r)$ and the upper bound $A(r) \le \widehat{A}(r)$.

We next show the lower bound for $A(r)$. Consider (independently) the following $r - 1$ colorings λ_j, $j = 2, \dots, r$. The coloring λ_j colors grid point (x, y) with color $(y \mod j)$. Observe that the area of a triangle with vertices (x_i, y_i), $i = 1, 2, 3$, is

$$\frac{|x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3|}{2}.$$

Let Δ be a monochromatic grid triangle of area $A(r)$ in this coloring. By symmetry, there is a congruent triangle Δ_0 of color 0, whose y-coordinates satisfy $y_1 \equiv y_2 \equiv y_3 \equiv 0 \pmod{j}$. Hence $2A(r)$ is a nonzero multiple of j. By repeating this argument for each j, we get that $2A(r)$ is a nonzero multiple of all numbers $2, \dots, r$, hence also of l.c.m. $(2, 3, \dots, r)$. This completes the proof of the lower bound.

We now prove that $\widehat{A}(r) \lll \widehat{T}(r)$. Although our estimates $\widehat{A}(r)$ also depend on the Van der Waerden numbers $W(k, r)$, the dependence shows a much more modest growth rate for $\widehat{A}(r)$ than for $\widehat{T}(r)$. For instance, since $W(3, 3) = 27$, we have $\widehat{A}(2) \le 13! \cdot 2! \approx 10^{10}$, while $\widehat{T}(2) \le 2W(5, 2)! = 2 \cdot 178! \approx 10^{325}$. We have only very imprecise estimates on Van der Waerden numbers available. The current best upper bound, due to Gowers [18], gives

$$W(k, r) \le 2^{2^{f(k,r)}}, \quad \text{where } f(k, r) = r^{2^{2^{k+9}}}.$$

In particular,

$$W(7, 7) \le 2^{2^{7^{2^{2^{16}}}}}, \text{ and } \widehat{A}(3) = \left\lfloor \frac{W(7, 7) - 1}{6} \right\rfloor ! \cdot 6\,.$$

On the other hand, Graham's estimate

$$\widehat{T}(3) = 2 \cdot 178!(2 \cdot 178! + 1) \cdot W(2 \cdot (2 \cdot 178! + 1)! + 1, 3)!$$

appears to be much larger.

The ratio between the two estimates amplifies even more for larger values of r. Let now $r \ge 4$. We have

$$\begin{aligned} \widehat{A}(r) = H! \cdot r! \;&=\; \left\lfloor \frac{W(r! + 1, 2^r - 1) - 1}{r!} \right\rfloor ! \cdot r! \\ &\le\; W(r! + 1, 2^r - 1)!\,. \end{aligned}$$

Write $\log^{(i)} x$ for the ith iterated binary logarithm of x. Using again very rough approximations such as

$$r! + 10 \le 2^{2^{r-1}} \quad \text{and} \quad 2^{2^{2^{2^{r-1}}}} \cdot r \le 2^{2^{2^{2^r}}},$$

we obtain

$$\begin{aligned} &\log^{(2)} \widehat{A}(r) \le W(r! + 1, 2^r - 1), \\ &\log^{(2)} W(r! + 1, 2^r - 1) \le f(r! + 1, 2^r - 1), \\ &\log^{(1)} f(r! + 1, 2^r - 1) \le 2^{2^{r!+10}} \log(2^r - 1) \le 2^{2^{2^{2^r}}}, \\ &\log^{(4)} 2^{2^{2^{2^r}}} = r. \end{aligned}$$

It follows that

$$\log^{(9)} \widehat{A}(r) \le r. \tag{1}$$

On the other hand, even if we ignore the predominant factor $W(2(S_i + 1)! + 1, i + 1)!$ in the expression of S_{i+1} when estimating $\widehat{T}(r)$, the inequality $S_{i+1} \ge (S_i + 1)! \ge 2^{S_i}$ still implies that

$$\widehat{T}(r) \ge 2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}, \quad \text{a tower of } r \text{ 2s.} \tag{2}$$

By comparing the two inequalities (1) and (2), we conclude that $\widehat{A}(r) \le \widehat{T}(r)$ for $r \ge 12$, and that $\widehat{A}(r) \lll \widehat{T}(r)$ for sufficiently large r. This gives a partial answer to Graham's question[1] raised in the conclusion of his paper [12], and completes the proof of Theorem 4.

Finally, observe that we can replace $H!$ and $r!$ with the smaller numbers l.c.m. $(2, 3, \dots, H)$ and l.c.m. $(2, 3, \dots, r)$, respectively, and thereby obtain:

[1] The proof by Adhikari [1] gives an alternative upper bound $T(r) \le \widehat{T'}(r)$. The reader can check that the same tower of 2s expression in (2) is also a lower bound on his estimate $\widehat{T'}(r)$ for $r \ge 2$.

Corollary 5 *Let*

$$H' = \left\lfloor \frac{W(\text{l.c.m.}(2,3,\ldots,r)+1, 2^r-1)-1}{\text{l.c.m.}(2,3,\ldots,r)} \right\rfloor.$$

Then

$$\begin{aligned} A(r) &\geq \frac{1}{2} \times \text{l.c.m.}(2,3,\ldots,r) = e^{(1+o(1))r}, \text{ and} \\ A(r) &\leq \text{l.c.m.}(2,3,\ldots,H') \times \text{l.c.m.}(2,3,\ldots,r). \end{aligned}$$

It is an easy exercise to show that the above lower bound is tight for $r = 2$, that is, $A(2) = 1$. Consider two cases:

1. If the 2-coloring of $\mathbb{Z}^2$ follows a chess-board pattern, say point (x, y) is colored $(x+y) \mod 2$, then clearly there is a monochromatic triangle of area 1, for example the triangle with vertices $(0, 0)$, $(1, 1)$, and $(0, 2)$.

2. Otherwise, there are two adjacent points of the same color, say $(0, 0)$ and $(1, 0)$ of color 0. Suppose there is no monochromatic triangle of area 1. Then $(0, 2)$ and $(2, 2)$ would have color 1. Then $(0, 1)$ and $(2, 1)$ would have color 0. Then the triangle with vertices $(0, 0)$, $(0, 1)$, and $(2, 1)$ would have color 0 and area 1, a contradiction.

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