Improved Bounds on the Average Distance to the Fermat-Weber Center of a Convex Object

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Abstract

We show that for any convex object Q in the plane, the average distance between the Fermat-Weber center of Q and the points in Q is at least $4\Delta(Q)/25$, and at most $2\Delta(Q)/(3\sqrt{3})$, where $\Delta(Q)$ is the diameter of Q. We use the former bound to improve the approximation ratio of a load-balancing algorithm of Aronov et al. [1].

1 Introduction

The Fermat-Weber center of an object Q in the plane is a point in the plane, such that the average distance from it to the points in Q is minimal. For an object Q and a point y, let $\mu_Q(y)$ be the average distance between y and the points in Q, that is, $\mu_Q(y) = \int_{x\in Q} \|xy\|\, dx/\text{area}(Q)$, where $\|xy\|$ is the Euclidean distance between x and y. Let $\mathcal{FW}_Q$ be a point for which this average distance is minimal, that is, $\mu_Q(\mathcal{FW}_Q) = \min_y \mu_Q(y)$, and put $\mu_Q^* = \mu_Q(\mathcal{FW}_Q)$. The point $\mathcal{FW}_Q$ is a Fermat-Weber center of Q.

It is easy to verify, for example, that the Fermat-Weber center of a disk D coincides with the center o of D, and that the average distance between o and the points in D is $\Delta(D)/3$, where $\Delta(D)$ is the diameter of D. Carmi, Har-Peled, and Katz [3] studied the relation between μ_Q^* and the diameter of Q, denoted $\Delta(Q)$. They proved that there exists a constant c_1, such that, for any *convex* object Q, the average distance between a Fermat-Weber center of Q and the points in Q is at least $c_1\Delta(Q)$, and that the largest such constant c_1^* lies in the range $[1/7..1/6]$.

In this paper, we both improve the above bound on c_1^*, and tightly bound a new constant c_2^*; see below. More precisely, we first significantly narrow the range in which c_1^* must lie, by proving (in Section 2) that $4/25 \le c_1^* \le 1/6$. Next, we consider the question what is the smallest constant c_2^*, such that, for any convex object Q, $\mu_Q^* \le c_2^*\Delta(Q)$. We prove (in Section 3) that $1/3 \le c_2^* \le 2/(3\sqrt{3})$. A useful corollary obtained from these results is that the average distance to the center of the smallest enclosing circle of a convex n-gon P is less than 2.41 times μ_P^*.

The Fermat-Weber center of an object Q is a very significant point. The classical Fermat-Weber problem is: Find a point in a set F of feasible facility locations, that minimizes the average distance to the points in a set D of (possibly weighted) demand locations. If D is a finite set of points, F is the entire plane, and distances are measured using the L_2 metric, then it is known that the solution is algebraic [2]. See Wesolowsky [8] for a survey of the Fermat-Weber problem.

Only a few papers deal with the continuous version of the Fermat-Weber problem, where the set of demand locations is continuous. Fekete, Mitchell and Weinbrecht [4] presented algorithms for computing an optimal solution for $D = F = P$ where P is a simple polygon or a polygon with holes, and the distance between two points in P is the L_1 geodesic distance between them. Carmi, Har-Peled and Katz [3] presented a linear-time approximation scheme for the case where P is a convex polygon.

Aronov et al. [1] considered the following load balancing problem. Let D be a convex region and let $\mathcal{P} = \{p_1, \ldots, p_m\}$ be a set of m points representing m facilities. One would like to divide D into m equal-area subregions $R_1, \ldots, R_m$, so that region R_i is associated with point p_i, and the total cost of the subdivision is minimized. Given a subdivision, the cost $\kappa(p_i)$ associated with facility p_i is the average distance between p_i and the points in R_i, and the total cost of the subdivision is $\sum_i \kappa(p_i)$.

Aronov, et al. discussed the structure of an optimal subdivision, and also presented an $(8 + \sqrt{2\pi})$-approximation algorithm, under the assumption that the regions $R_1, \ldots, R_m$ must be convex and that D is a rectangle. Our improved bound on the constant c_1^*, allows us (in Section 4) to improve the above approximation ratio.

2 $4/25 \le c_1^* \le 1/6$

Carmi, Har-Peled and Katz [3] showed that there exists a convex polygon P such that $\mu_P^* \le \Delta(P)/6$. This immediately implies that $c_1^* \le 1/6$. We prove below that $c_1^* \ge 4/25$. Our proof is similar in its structure to the proof of [3].

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Theorem 2.1. *Let P be a convex object. Then $\mu_P^* \geq 4\Delta(P)/25$.*

Proof: Let $\mathcal{FW}_P$ be a Fermat-Weber center of P. We need to show that $\int_{x\in P} \|x\mathcal{FW}_P\|\,dx \geq \frac{4\Delta(P)}{25}\text{area}(P)$. We do this in two stages. In the first stage we show that for a certain subset P' of P, $\int_{x\in P'} \|x\mathcal{FW}_P\|\,dx \geq \frac{4\Delta(P)}{27}\text{area}(P)$. This implies that for any convex object Q, $\mu_Q^* \geq 4\Delta(Q)/27$. In the second stage we apply this intermediate result to a collection of convex subsets of $P - P'$ that are pairwise disjoint to obtain the claimed result. This latter stage is essentially identical to the second stage in the proof of [3]; it is included here for the reader's convenience.

We now describe the first stage. Let s be a line segment of length $\Delta(P)$ connecting two points p and q on the boundary of P. We may assume that s is horizontal and that p is its left endpoint, since one can always rotate P around, say, p until this is the case.

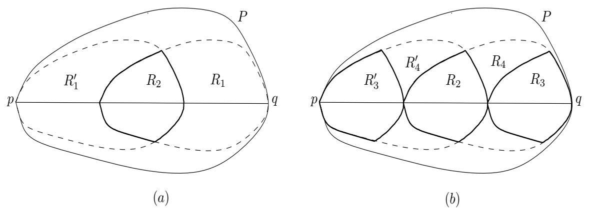

Figure 1: Proof of intermediate result.

Let P^α be the polygon obtained from P by shrinking it by a factor of α, that is, by applying the transformation $f(a,b) = (a/\alpha, b/\alpha)$ to the points (a,b) in P. We place a copy R_1 of $P^{3/2}$, such that R_1 is contained in P and has a common tangent with P at q. Similarly, we place a copy R'_1 of $P^{3/2}$, such that R'_1 is contained in P and has a common tangent with P at p; see Figure 1(a). Clearly, $\text{area}(R_1) = \text{area}(R'_1) = \frac{4}{9}\text{area}(P)$.

Let $R_2 = R_1 \cap R'_1$. We place a copy R_3 of R_2, such that R_3 is contained in R_1 and has a common tangent with R_1 at q. Similarly, we place a copy R'_3 of R_2, such that R'_3 is contained in R'_1 and has a common tangent with R'_1 at p. Let $R_4 = R_1 - (R_2 \cup R_3)$ and $R'_4 = R'_1 - (R_2 \cup R'_3)$; see Figure 1(b).

We know that, regardless of the exact location of $\mathcal{FW}_P$, the distance between $\mathcal{FW}_P$ and the points in R_3 plus the distance between $\mathcal{FW}_P$ and the points in R'_3 is greater than $\frac{2\Delta(P)}{3}\text{area}(R_3)$, and the distance between $\mathcal{FW}_P$ and the points in R_4 plus the distance between $\mathcal{FW}_P$ and the points in R'_4 is greater than $\frac{\Delta(P)}{3}\text{area}(R_4)$. More precisely,

$$\int_{x\in R_3} \|x\mathcal{FW}_P\|\,dx + \int_{x\in R'_3} \|x\mathcal{FW}_P\|\,dx \geq \frac{2\Delta(P)}{3}\text{area}(R_3)$$

and

$$\int_{x\in R_4} \|x\mathcal{FW}_P\|\,dx + \int_{x\in R'_4} \|x\mathcal{FW}_P\|\,dx \geq \frac{\Delta(P)}{3}\text{area}(R_4)\,.$$

Since $\text{area}(R_4) = \text{area}(R_1) - (\text{area}(R_2) \cup \text{area}(R_3)) = \frac{4}{9}\text{area}(P) - 2\text{area}(R_3)$, we obtain our intermediate result

$$\begin{aligned}\int_{x\in P} \|x\mathcal{FW}_P\|\,dx &\geq \int_{x\in R_3} \|x\mathcal{FW}_P\|\,dx + \\ &+ \int_{x\in R'_3} \|x\mathcal{FW}_P\|\,dx + \int_{x\in R_4} \|x\mathcal{FW}_P\|\,dx + \\ &+ \int_{x\in R'_4} \|x\mathcal{FW}_P\|\,dx \geq \frac{2\Delta(P)}{3}\text{area}(R_3) + \\ &+ \frac{\Delta(P)}{3}\left(\frac{4}{9}\text{area}(P) - 2\text{area}(R_3)\right) = \frac{4\Delta(P)}{27}\text{area}(P)\,.\end{aligned}$$

This intermediate result immediately implies that for any convex object Q, $\mu_Q^* \geq 4\Delta(Q)/27$. In the second stage we show that the 27 in the denominator can be replaced by 25.

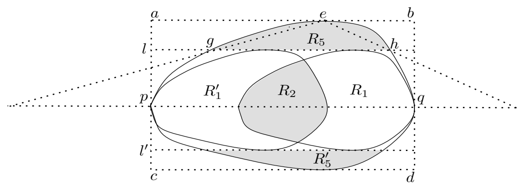

Figure 2: Proof of improved result.

Consider Figure 2. We draw the axis-aligned bounding box of P. The line segment s (whose length is $\Delta(P)$) divides the bounding box of P into two rectangles, $abqp$ above s and $pqdc$ below s. We divide each of these rectangles into two parts (a lower part and an upper part), by drawing the two horizontal lines l and l'. Let R_5 denote the intersection of P with the upper part of the upper rectangle, and let R'_5 denote the intersection of P with the lower part of the lower rectangle.

Let e be any point on the segment ab that also lies on the boundary of R_5. We mention several facts concerning R_5 and R'_5. $R_5 \cap R'_5 = \phi$, $R_5 \cap R_1 = \phi$, $R_5 \cap R'_1 = \phi$, $R'_5 \cap R_1 = \phi$, and $R'_5 \cap R'_1 = \phi$. Notice also that $\Delta(R_5)$, $\Delta(R'_5) \geq \Delta(P)/3$, since, e.g., the line segment $l \cap R_5$ contains the base of the triangle that is obtained by intersecting the triangle peq with R_5, and the length of this base is $\Delta(P)/3$.

We observe that $\text{area}(R_5) + \text{area}(R'_5) \geq \text{area}(P)/9$ by showing that $\text{area}(R_5) \geq \text{area}(P \cap abqp)/9$ (and that $\text{area}(R'_5) \geq \text{area}(P \cap pqdc)/9$). Let g, h be the two points on the line l that also lie on the boundary of R_5. Let $l(s)$ be the line containing s, and let T be the triangle defined by $l(s)$ and the two line segments connecting e to $l(s)$ and passing through g and through h, respectively. Let T_2 denote the triangle geh.

Clearly $T_2 \subseteq R_5$. Put $Q = R_5 - T_2$. Then, $\text{area}(R_5) = \text{area}(T_2) + \text{area}(Q) = \text{area}(T)/9 + \text{area}(Q)$. Therefore, $\text{area}(R_5) \geq (\text{area}(T) + \text{area}(Q))/9 \geq \text{area}(P \cap abqp)/9$. We show that $\text{area}(R_5') \geq \text{area}(P \cap pqdc)/9$ using the "symmetric" construction. Since $(P \cap abqp) \cup (P \cap pqdc) = P$ we obtain that $\text{area}(R_5) + \text{area}(R_5') \geq \text{area}(P)/9$.

It is also easy to see that $\Delta(R_2) = \Delta(P)/3$ and $\text{area}(R_2) \geq \text{area}(P)/9$. This is because $P^3 \subseteq R_2$ and $\text{area}(P^3) = \text{area}(P)/9$, where P^3 is the polygon obtained from P by shrinking it by a factor of 3.

Now using the implication of our intermediate result we have

$$\begin{aligned}
&\int_{x \in R_5} \|x\mathcal{FW}_P\|\, dx + \int_{x \in R_5'} \|x\mathcal{FW}_P\|\, dx + \\
&\quad + \int_{x \in R_2} \|x\mathcal{FW}_P\|\, dx \geq \frac{4\Delta(R_5)}{27}\text{area}(R_5) + \\
&\quad + \frac{4\Delta(R_5')}{27}\text{area}(R_5') + \frac{4\Delta(R_2)}{27}\text{area}(R_2) \geq \\
&\quad \geq \frac{4\Delta(P)}{81}(\text{area}(R_5) + \text{area}(R_5') + \text{area}(R_2)) \geq \\
&\quad \geq \frac{8\Delta(P)}{729}\text{area}(P)\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_{x \in P} \|x\mathcal{FW}_P\|\, dx \geq \int_{x \in R_3} \|x\mathcal{FW}_P\|\, dx + \\
&\quad + \int_{x \in R_3'} \|x\mathcal{FW}_P\|\, dx + \int_{x \in R_4} \|x\mathcal{FW}_P\|\, dx + \\
&\quad + \int_{x \in R_4'} \|x\mathcal{FW}_P\|\, dx + \int_{x \in R_5} \|x\mathcal{FW}_P\|\, dx + \\
&\quad + \int_{x \in R_5'} \|x\mathcal{FW}_P\|\, dx + \int_{x \in R_2} \|x\mathcal{FW}_P\|\, dx \geq \\
&\quad \geq \frac{4\Delta(P)}{27}\text{area}(P) + \frac{8\Delta(P)}{729}\text{area}(P) = \\
&\quad = \frac{116\Delta(P)}{729}\text{area}(P)\,.
\end{aligned}$$

At this point we may conclude that for any convex object Q, $\mu_Q^* \geq 116\Delta(Q)/729$. So we repeat the calculation above using this result for the regions R_5, R_5' and R_2 (instead of using the slightly weaker result, i.e., $\mu_Q^* \geq 4\Delta(Q)/27$). This calculation will yield a slightly stronger result, etc. In general, the result after the k-th iteration is $\mu_Q^* \geq c_k\Delta(Q)$, where $c_k = 4/27 + 2c_{k-1}/27$ and $c_0 = 4/27$. It is easy to verify that this sequence of results converges to $\mu_Q^* \geq 4\Delta(Q)/25$. ■

Corollary 2.2. *Let P be a non-convex simple polygon, such that the ratio between the area of a minimum-area enclosing ellipse of P and the area of a maximum-area enclosed ellipse is at most β, for some constant $\beta \geq 1$. Then $\mu_P^* \geq 4\Delta(P)/(25\beta^2)$.*

Proof: As in [3], except that we apply the improved bound of Theorem 2.1. ■

3 $1/3 \leq c_2^* \leq 2/(3\sqrt{3})$

As mentioned in the introduction, the average distance between the Fermat-Weber center of a disk D (i.e., D's center) and the points in D is $\Delta(D)/3$, where $\Delta(D)$ is the diameter of D. This immediately implies that $c_2^* \geq 1/3$. We prove below that $c_2^* \leq 2/(3\sqrt{3})$.

We first state a simple lemma and a theorem of Jung that are needed for our proof.

Lemma 3.1. *Let R, Q be two (not-necessarily convex) disjoint objects, and let p be a point in the plane. Then, $\mu_{(R\cup Q)}(p) \leq \max\{\mu_R(p), \mu_Q(p)\}$.*

Proof:

$$\begin{aligned}
\mu_{(R\cup Q)}(p) &= \frac{\int_{x\in R\cup Q}\|px\|\,dx}{\text{area}(R\cup Q)} = \\
&= \frac{\int_{x\in R}\|px\|\,dx + \int_{x\in Q}\|px\|\,dx}{\text{area}(R) + \text{area}(Q)} = \\
&= \frac{\text{area}(R)\cdot\mu_R(p) + \text{area}(Q)\cdot\mu_Q(p)}{\text{area}(R) + \text{area}(Q)} \leq \\
&\leq \frac{(\text{area}(R) + \text{area}(Q))\max\{\mu_R(p), \mu_Q(p)\}}{\text{area}(R) + \text{area}(Q)} \leq \\
&\leq \max\{\mu_R(p), \mu_Q(p)\}\ .
\end{aligned}$$

■

Theorem 3.2 (Jung's Theorem [5, 6]). *Every set of diameter d in $\mathbb{R}^n$ is contained in a closed ball of radius $r \leq d\sqrt{\frac{n}{2(n+1)}}$. In particular, if R is a convex object in the plane, then the radius of the smallest enclosing circle C of R is at most $\Delta(R)/\sqrt{3}$, where $\Delta(R)$ is the diameter of R.*

Theorem 3.3. *For any convex object R, $\mu_R^* \leq 2\Delta(R)/(3\sqrt{3})$.*

Proof: Let R be a convex polygon. Let C be the smallest enclosing circle of R, and let o and r denote R's center point and radius, respectively. Notice that $o \in R$, since R is convex. We divide R into 8 regions $R_1, \ldots, R_8$ by drawing four line segments through o, such that each of the 8 angles formed around o is of 45°; see Figure 3(a). Clearly, for each R_i, $o \in R_i$ and $\Delta(R_i) \leq r$.

We first prove that for each region R_i, $\mu_{R_i}(o) \leq 2\Delta(R_i)/3$. (This is done by adapting the proof of Lemma 3.1 of Aronov et al. [1].) Consider Figure 3(b). Let $p \in R_i$ be the farthest point from o. Draw the circular sector ocd centered at o of radius $\|op\|$. Let a and b be as in Figure 3(b). Let f be the point on the arc cd, such that the regions Q_1 and Q_2 obtained by drawing the segment of are of equal area. (Q_1 is the region oxb and Q_2 is the difference between the sector opf and the

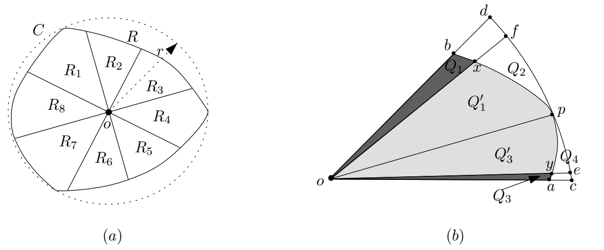

Figure 3: Illustrating the proof of Theorem 3.3.

region opx, where x is the intersection point between of and the boundary piece pb.) Similarly, let e be the point on the arc cd, such that the regions Q_3 and Q_4 obtained by drawing the segment oe are of equal area. (Q_3 is the region oay and Q_4 is the difference between the sector oep and the region oyp, where y is the intersection point between oe and the boundary piece ap.)

Now, on the one hand, since opb is convex, x is the farthest point from o in Q_1, and, on the other hand, x is the closest point to o in Q_2. Hence, any point in Q_2 is farther from o than any point in Q_1. Thus we get that $\mu_{opb}(o) = \mu_{(Q'_1 \cup Q_1)}(o) \leq \mu_{(Q'_1 \cup Q_2)}(o) = \mu_{opf}(o) = 2\,\|op\|/3 = 2\Delta R_i/3$. We show that $\mu_{oap}(o) \leq 2\Delta R_i/3$ using the "symmetric" analysis. Since opb and oap are disjoint convex objects, then, by Lemma 3.1, $\mu_{R_i}(o) = \mu_{(opb \cup oap)}(o) \leq 2\Delta R_i/3$.

We now show that $\mu_R(o) \leq 2\Delta(R)/(3\sqrt{3})$, immediately implying that $\mu^*_R \leq 2\Delta(R)/(3\sqrt{3})$. By Theorem 3.2, we know that $r \leq \Delta(R)/\sqrt{3}$. We also know that for each R_i, $\Delta(R_i) \leq r$. Thus, $\mu_{R_i}(o) \leq 2\Delta(R_i)/3 \leq 2r/3 \leq 2\Delta(R)/(3\sqrt{3})$.

We now apply Lemma 3.1 to obtain that

$$
\begin{aligned}
\mu_R(o) &\leq \max\left\{\mu_{(R_1 \cup R_2 \cup R_3 \cup R_4)}(o), \mu_{(R_5 \cup R_6 \cup R_7 \cup R_8)}(o)\right\} \leq \\
&\leq \max\left\{\max\left\{\mu_{(R_1 \cup R_2)}(o), \mu_{(R_3 \cup R_4)}(o)\right\},\right. \\
&\qquad\left. \max\left\{\mu_{(R_5 \cup R_6)}(o), \mu_{(R_7 \cup R_8)}(o)\right\}\right\} \leq \\
&\ \ \vdots \\
&\leq \max\{\mu_{R_1}(o), \mu_{R_2}(o), \mu_{R_3}(o), \mu_{R_4}(o)\,, \\
&\qquad \mu_{R_5}(o), \mu_{R_6}(o), \mu_{R_7}(o), \mu_{R_8}(o)\} \leq \\
&\leq 2\Delta(R)/(3\sqrt{3})\,. \qquad \blacksquare
\end{aligned}
$$

Corollary 3.4. *Let P be a convex n-gon. Then one can compute in linear time a point p, such that $\mu_P(p) \leq \frac{25}{6\sqrt{3}}\mu^*_P$.*

Proof: We apply Megiddo's linear-time algorithm for computing the smallest enclosing circle C of P [7]. Let p denote the center of C, then, by Theorem 2.1

$$
\frac{\mu_P(p)}{\mu^*_P} \leq \frac{2\Delta(P)/(3\sqrt{3})}{4\Delta(P)/25} = \frac{25}{6\sqrt{3}}\,. \qquad \blacksquare
$$

Corollary 3.4 gives us a very simple linear-time constant-factor approximation algorithm for finding an approximate Fermat-Weber center in a convex polygon. A less practical linear approximation scheme for finding such a point was presented by Carmi et al. [3].

4 Application

We consider the load balancing problem studied by Aronov et al. [1]. Let D be a convex region and let $\mathcal{P} = \{p_1, \ldots, p_m\}$ be a set of m points representing m facilities. The goal is to divide D into m equal-area *convex* regions $R_1, \ldots, R_m$, so that region R_i is associated with point p_i, and the total cost of the subdivision is minimized. The cost $\kappa(p_i)$ associated with facility p_i is the average distance between p_i and the points in R_i, and the total cost of the subdivision is $\sum_i \kappa(p_i)$.

Assuming D is a rectangle that can be divided into m squares of equal size, Aronov et al. present an $O(m^3)$-time algorithm for computing a subdivision of cost at most $(8+\sqrt{2\pi})$ times the cost of an optimal subdivision. By applying Theorem 2.1 in the analysis of their algorithm, we obtain a better approximation ratio, namely, $(\frac{29}{4} + \sqrt{2\pi})$. For further details, see the full version of this paper.

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